United Kingdom Mathematics Trust

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Mathematical Olympiad for Girls 2019

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

Olympiad marking

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for is full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has a viable overall strategy or not. An answer which is essentially a solution will be awarded near maximum credit, with marks deducted for errors of calculation, flaws in logic, omission of cases or technical faults. One question we often ask is: if we were to have the benefit of a two-minute interview with this candidate, could they correct the error or fill the gap? On the other hand, an answer which does not present a complete argument is marked on a '0 plus' basis; up to 4 marks might be awarded for particular cases or insights.

This approach is therefore rather different from what happens in public examinations such as GCSE, AS and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore important that candidates taking Olympiad papers realise the importance of the comment in the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

General comments

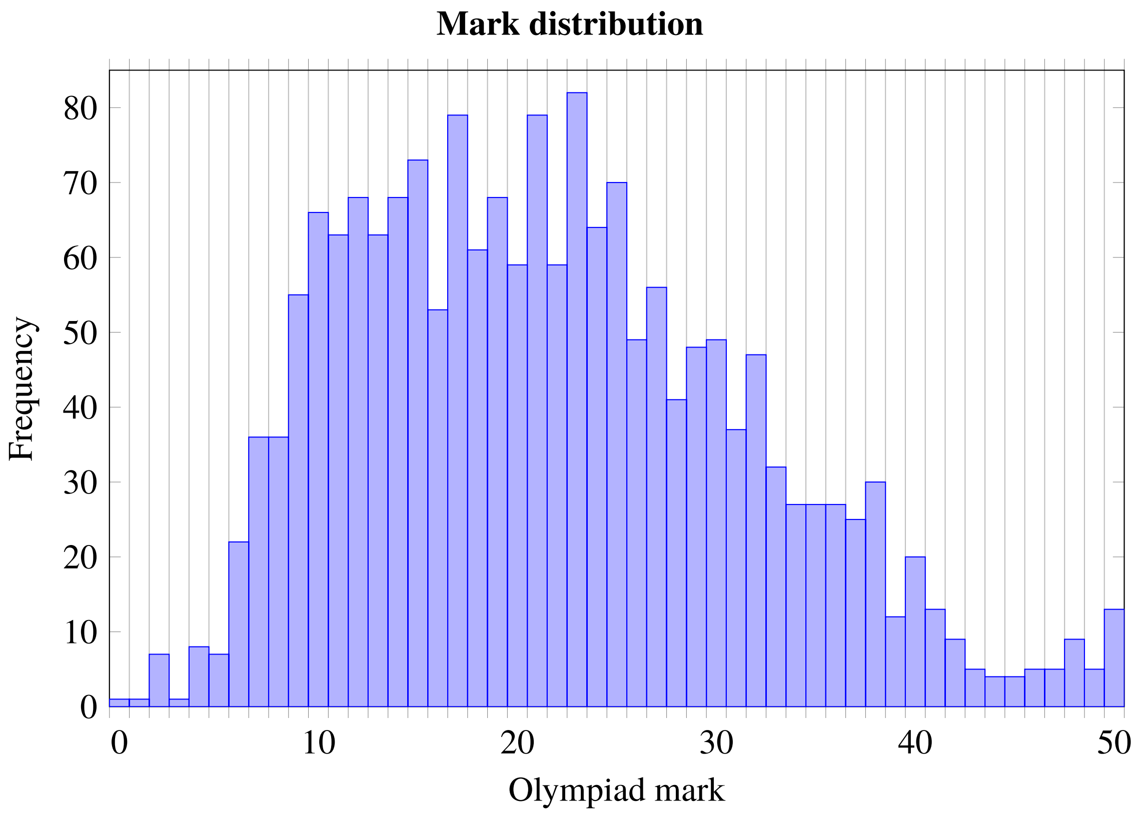
Students engaged very well with this paper and the markers enjoyed reading many imaginative and well explained solutions. A vast majority of candidates attempted all the questions and were often able to score at least one or two marks even on the later ones. Over two-thirds of the candidates solved at least one question fully.

In last year's report we commented on the need to consider part (a) of a question as a hint for doing part (b). This advice was followed well this year, particularly in Questions 2 and 3 where we saw many good attempts to use a colouring argument, or to find overlaps between sets of multiples. Even in Question 5, a record number of candidates made progress in part (b) by trying to pair up the lists in some way, as suggested by part (a).

There were several common ways to lose marks. Questions 1 and 4 both asked for a maximum possible value of a quantity; often the students found the required value but did not justify why it is a maximum, or show that it can in fact be achieved within the constraints of the question. In Questions 1, 2 and 5 it was sensible to start by trying some examples (possible combinations of cats and dogs; first few moves of the counters; continuing Tracy's lists). However, those examples can only suggest a pattern or rule that then needs to be explained and justified. It was somewhat disappointing to see algebraic mistakes in Questions 1 and 4 when dealing with fractions and square roots.

Overall, however, the quality of candidates was very high. We were particularly impressed by solutions to Question 3, which produced the largest number of well-explained, structured arguments.

The 2019 Mathematical Olympiad for Girls attracted 1848 entries. The scripts were marked on 19th and 20th October in Cambridge by a team of Amelia Rout, Amit Shah, Anđela Šarković, Andreas Stavrou, Andrew Carlotti, Arij Asad, Abhilasha Aggarwal, Emily Beatty, Eve Pound, James Cranch, Jeremy King, Jerome Watson, Joseph Myers, Kasia Warburton, Lulu Beatson, Martin Orr, Melissa Quail, MT Fyfe, Oscar Hidalgo, Philip Coggins, Phillip Beckett, Robin Bhattacharyya, Stephen Tate, Vesna Kadelburg and Zarko Randjelovic.


Mark distribution
Frequency
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Olympiad mark

Question 1

At Mathsland Animal Shelter there are only cats and dogs. Unfortunately, one day 60 of the animals managed to escape. Once a volunteer had realised, they counted the remaining animals. They noted that half of the cats and a third of the dogs had escaped.

(a) (i) If the number of cats before the escape was C and the number of dogs before the escape was D, write down an equation linking C and D.

(ii) If the total number of animals before the escape was T, write down an equation linking C, D and T.

(4 marks)

(b) Given that more cats than dogs escaped, find the largest possible value of T. You must justify why the value you have found is the largest. (6 marks)

Solution

(a) The total number of escaped animals is 60, and the total number of animals before the escape is T.

(i) $\frac{1}{2}C + \frac{1}{3}D = 60$

(ii) $C + D = T$

(b) From the first equation, $D = 180 - \frac{3}{2}C$. Substituting into the second equation,

$$T = C + (180 - \tfrac{3}{2}C) = 180 - \tfrac{1}{2}C.$$

Therefore T is the largest possible when C is the smallest possible. Since more half of the 60 escaped animals were cats, $\frac{1}{2}C \geq 31$, so $T \leq 180 - 31 = 149$. This value is achieved with $C = 62$ and $D = 87$.

Hence the largest possible value of T is 149.

Alternative

We start as above, but express $\frac{1}{2}C$ in terms of T: $\frac{1}{2}C = 180 - T$. We can similarly find $\frac{1}{3}D = T - 120$.

Substituting those two into $\frac{1}{2}C > \frac{1}{3}D$ gives$180 - T > T - 120$, which rearranges to $T < 150$. Thus the largest possible value of T is 149, which can be achieved with $C = 62$ and $D = 87$.

Markers' comments

There were many very good solutions to this problem, usually following one of the strategies presented above. In the first solution, it was important to explain both why $\frac{1}{2}C \geq 31$ (or $\frac{1}{3}D \leq 29$), and also why the greatest possible value of T requires the greatest possible value of D (or the smallest possible value of C).

Noting that $\frac{1}{2}C \geq 31$ and $\frac{1}{3}D \leq 29$ does not on its own explain why taking $\frac{1}{2}C = 31$ and $\frac{1}{3}D = 29$ will produce the maximum value of T. Just being at one end of a region of possible

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values for C or D does not guarantee a maximum value for T. Several students assumed it did (or gave a few numerical calculations to support this proposition) – they didn't get credit for solving the problem. It was vital to give a logical argument for why T would be maximised.

One way to justify this is as shown in the the first solution above. An alternative justification, favoured by a majority of candidates, was to say that, in finding T from $\frac{1}{2}C$ and $\frac{1}{3}D$, the value of $\frac{1}{2}C$ is doubled but the value of $\frac{1}{3}D$ is tripled; hence we want $\frac{1}{3}D$ to be as large as possible.

The alternative solution above proves that 149 is the maximum value of T, but then it needs to be shown that this value can actually be achieved by stating which values of C and D should be used.

Some candidates used a graphical method, looking at the maximum value of $C + D$ on the part of the line $\frac{1}{2}C + \frac{1}{3}D = 60$ which lies in the region $\frac{1}{2}C > \frac{1}{3}D$. It is again important to explain which end of the allowed region gives the maximum value of $C + D$.

Question 2

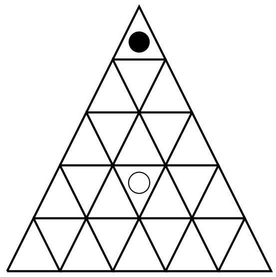
Beth has a black counter and Wendy has a white counter. Beth and Wendy move their counters on the two boards below according to the starting positions and rules given. They always move their counters at the same time.

(a) 

At each turn, each player moves their counter either one square to the left or one square to the right. Prove that the black and white counters can never be in the same square at the same time.

Hint You may find it helpful to refer to the colours of the squares on the board in your explanation. (3 marks)

(b) At each turn, each player moves their counter to a triangular cell which shares one edge with the cell that their counter is currently in. Can their counters ever be in the same cell at the same time? (7 marks)

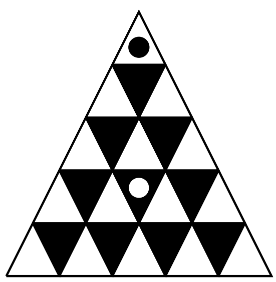


Hint If you think the two counters can never be in the same cell at the same time, you should give an argument that they cannot be in the same cell at the same time which works no matter which sequence of moves Beth and Wendy do. If you think the two counters can be in the same cell at the same time, you should give an example of a sequence of moves after which they are in the same cell at the same time.

Solution

(a) Since the colours of the squares on the board alternate, each move (one square to the left or one square to the right) changes the colour of the square that the counter is in. The two counters start on different colours so, since each one changes the colour at each turn, they will always be on different colours. This means that they can never be in the same square at the same time.

(b) Colour the triangular cells black and white as shown in the diagram.



Then two cells which share an edge are different colours. Hence each move changes the colour of the counter's cell.

As can be seen in the diagram, the black and the white counter start on different colours. Therefore they will always be on different colours, and so cannot be in the same cell at the same time.

Markers' comments

A large number of candidates gained all 3 marks in part (a), however some students lost marks due to vague statements that didn't demonstrate a clear understanding of the key elements of the argument. In general, a good colouring argument will have 3 main parts:

- A statement of the initial position (e.g. "Counters start on opposite colours.")
- A description of what happens at each turn (e.g. "The colour each counter is on switches every turn from white to black or black to white.")
- A conclusion about what this means for the final position (e.g. "So the counters are always on opposite colours to each other and therefore can't be in the same square at the same time.")

In particular, many candidates commented on the change of colour on the first move, but didn't clearly explain how and why this continues.

Part (a) proved to be a useful hint for part (b), where the markers were pleased to see many good arguments referring to either the orientation of the triangular cells ("up" and "down"), or a colouring of the board as shown in our solution. It was again important to explain explicitly why the colour (or orientation) changes on each turn.

Many candidates managed to complete part (a) successfully by considering the parity of the distance between the two counters rather than by directly using the colours of the squares. However, they often went on to attempt to complete part (b) via a 'distance between the two counters' argument. The 'distance' between two triangular cells is difficult to properly define since there are multiple routes between any pair of triangles. Even if you consider the 'shortest distance' it is hard to correctly justify that this will always have the same parity since it is not obvious that each move changes the shortest distance to a fixed cell by exactly one.

However, there is a very good argument along these lines which does work, and which was found by at least one student. We can count how many of the (long) straight lines in the diagram, running in any of the three directions, pass between the two counters in question. Then it is clear that when a counter moves by one space it will change the number of lines in between it and the other fixed cell by exactly one.

Question 3

(a) Seth wants to know how many positive whole numbers from one to one hundred are divisible by two or five. He thinks that the answer is 70 because there are fifty multiples of two and twenty multiples of five from one to one hundred. Explain why his answer is too large. (2 marks)

(b) Consider the list of 1800 fractions

$$\frac{1}{1800}, \frac{2}{1800}, \ldots, \frac{1799}{1800}, \frac{1800}{1800}.$$

How many are *not* in simplest form? Explain your reasoning. (8 marks)

[Note: The fraction $\frac{900}{1800}$ is not in simplest form because it can be simplified to $\frac{1}{2}$.]

SOLUTION

(a) If we list all 50 multiples of 2 and all 20 multiples of 5, all the multiples of 10 will appear in both lists. Since some numbers are counted twice, the real answer is smaller than 70.

(b) A fraction will not be in simplest form when the numerator shares at least one factor with 1800. Since $1800 = 2^3 \times 3^2 \times 5^2$, we need to count how many of the numbers from 1 to 1800 are divisible by 2, 3 or 5.

Imagine making three separate lists: one containing the multiples of 2, one containing the multiples of 3 and one containing the multiples of 5. There are $\frac{1800}{2} = 900$ numbers in the first list, $\frac{1800}{3} = 600$ in the second and $\frac{1800}{5} = 360$ in the third, a total of 1860 numbers.

The multiples of 6, 10 and 15 appear twice, so we need to take away $\frac{1800}{6} + \frac{1800}{10} + \frac{1800}{15} = 600$.

However, the multiples of 30 have now been taken away three times. But they appear three times in the original three lists, so they should have only been taken away twice. Hence we need to add back $\frac{1800}{30} = 60$.

The required total is therefore $1860 - 600 + 60 = 1320$. Thus 1320 of the fractions are not in simplest form.

NOTE

The method in the above solutions (taking away overlaps, then adding back "overlaps of overlaps") is known as the inclusion-exclusion principle. It can be nicely illustrated on a Venn diagram and generalises to more than three sets. The simplest case of it, for two sets, was used in part (a).

ALTERNATIVE

We can count the numbers which are multiples of at least one of 2, 3 or 5 in the following way.

There are $\frac{1800}{2} = 900$ even numbers. Of the remaining 900 odd numbers, every third one is a multiple of 3, so we need to add $\frac{900}{3} = 300$ multiples of 3. This leaves $900 - 300 = 600$

odd numbers which are not multiples of three. Of those, every fifth one is a multiple of 5, giving an extra $\frac{600}{5} = 120$ multiples of 5 which have not yet been counted.

Hence the required number is $900 + 300 + 120 = 1320$.

Markers' comments

This was the most successful question on the paper in terms of the number of candidates finding a good solution strategy for part (b). The markers were very pleased with the number of scripts that picked up on the hint from part (a) and showed a good understanding of how to deal with at least some of the "overlaps".

The key to solving this problem fully was to extend the idea to the overlap of all three sets of multiples. A pleasingly large number of candidates realised that this needed to be done, but the difficult bit was deciding whether the multiples of 30 needed to be taken away or added back. A Venn diagram often proved a successful way of visualising the problem.

Many students proceeded along the lines of the alternative solution above and were often successful. The most common mistake was to find a fifth of the 300 odd multiples of 3, rather than a fifth of the remaining 600 odd numbers (as the multiples of three have already been accounted for).

Some students tried to find how many numbers from 1 to 100 inclusive, for example, share a prime factor with 1800. But 100 is not divisible by 3, and this means that the numbers from 101 to 200 and from 201 to 300 do not behave in the same way for this problem as the numbers from 1 to 100. So things don't 'scale up' properly. What would work, instead, would be to take the numbers from 1 up to some number that is a multiple of 2, of 3 and of 5. For example, we could look at the numbers from 1 to 30. This will then 'scale up' properly to the numbers from 1 to 1800. We have 22 numbers from 1 to 30 that are divisible by at least one of 2, 3 or 5, and then we get $22 \times \frac{1800}{30} = 1320$ such numbers from 1 to 1800, which is the correct answer to the whole problem.

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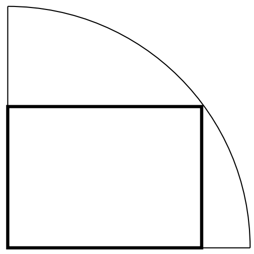
Question 4

The diagram shows a rectangle placed inside a quarter circle of radius 1, such that its vertices all lie on the perimeter of the quarter circle and one vertex coincides with the centre of the (whole) circle.



Let the perimeter of such a rectangle be P.

(a) Show that $P = 3$ is impossible. (4 marks)

(b) Find the largest possible value of P. You must fully justify why the value that you find is the largest. (4 marks)

Instead a rectangle is placed inside a whole circle of radius 1, such that its vertices all lie on the circumference of the circle.

(c) If the perimeter of the rectangle is as large as possible, show that the rectangle *must* be a square and calculate its perimeter. (2 marks)

Solution

Let x and y be the sides of the rectangle. The diagonal of the rectangle is a radius of the circle, so $x^2 + y^2 = 1$. The perimeter of the rectangle is $P = 2x + 2y$. Substituting $y = \dfrac{P - 2x}{2}$ from the second equation into the first gives

$$x^2 + \left(\frac{P - 2x}{2}\right)^2 = 1,$$

which is equivalent to

$$8x^2 - (4P)x + (P^2 - 4) = 0.$$

(a) When $P = 3$ this quadratic equation becomes $8x^2 - 12x + 5 = 0$. The discriminant is $12^2 - 4 \times 8 \times 5 = -16 < 0$ so there are no solutions for x. It is therefore not possible that $P = 3$.

(b) We now look for values of P for which the quadratic equation $8x^2 - (4P)x + (P^2 - 4) = 0$ has a solution. The discriminant needs to be non-negative, so we need

$$(4P)^2 - 32(P^2 - 4) \geq 0.$$

This is equivalent to $16P^2 \leq 128$ and, since $P > 0$, we must have $P \leq 2\sqrt{2}$. For this value of P, solving the quadratic equation gives $x = \frac{\sqrt{2}}{2}$ and, substituting back into $2x + 2y = 2\sqrt{2}$, $y = \frac{\sqrt{2}}{2}$. Thus it is possible that $P = 2\sqrt{2}$ and this is the largest possible value of P.

(c) Let the sides of the rectangle be $2x$ and $2y$. The diagonal of the rectangle is a diameter of the circle, so $x^2 + y^2 = 1$. (Note that this is the same relationship between x and y as in part (b).) The perimeter of the rectangle is $4x + 4y = 2(2x + 2y)$, which is twice the perimeter from part (b).

We know from part (b) that the largest possible value of $2x + 2y$ is $2\sqrt{2}$, and that this value only occurs when $x = y = \frac{\sqrt{2}}{2}$. Therefore the largest possible value of our perimeter is $4\sqrt{2}$ and it occurs when the sides of the rectangle are equal, i.e. when it is a square.

MARKERS' COMMENTS

This may look like a geometry question, but it is in fact much more an algebra question.

Writing down equations to represent the information in the question is the first important step. Many students who in part (a) wrote out both the circle equation $x^2 + y^2 = 1$ and also the perimeter condition as the equation $2x + 2y = 3$ were then able to substitute one equation into the other to get a quadratic equation in one variable. Many of these students then knew to use the discriminant to show that there were no solutions to this; some used the full quadratic formula instead, and a few students completed the square.

Many students who solved part (a) in this way didn't see that forming the perimeter equation with P instead of 3 would lead to a very similar solution for part (b). Many students did notice this, and solved part (b) in this way.

Some students, though, using whatever method, solved both parts together, as proving (b) and noting that $2\sqrt{2} < 3$ would also solve (a). Indeed, a good approach to (b) could earn some marks in both parts (a) and (b), even if otherwise a student's attempt at part (a) didn't gain a mark.

There were several other methods that would actually work for this problem (for both parts (a) and (b)), and it was heartening to see that students found many such methods in the exam. Some students used calculus to maximise the perimeter, differentiating the perimeter expression $2x + 2\sqrt{1 - x^2}$ or instead $2\cos\theta + 2\sin\theta$. However, many of these candidates did not check that their stationary point was actually a maximum point (for example by checking the second derivative).

Quite a few students drew a graph of the circle $x^2 + y^2 = 1$ and the straight line $x + y = \frac{P}{2}$ and then realised that the line which is tangent to the circle (in the first quadrant) would give the greatest value of P – but some understanding of the fact that larger values of P correspond to lines further up and to the right on the graph needed to be conveyed. Also, a symmetry argument, or the radius-tangent theorem, would explain why the point of tangency is also on the line $y = x$, and so can be calculated to be the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$, giving the maximum possibility for the perimeter as $2\sqrt{2}$.

Another approach used was to square the perimeter expression $2\cos\theta + 2\sin\theta$ and then use trigonometric identities to discover that this square is equal to $4 + 4\sin 2\theta$ which is at most 8: so the perimeter is at most $\sqrt{8}$.

But claiming just from the shape of the graphs of $y = \sin\theta$ and $y = \cos\theta$ that the maximum of the perimeter $2\cos\theta + 2\sin\theta$ will occur when θ is equal to 45 degrees did not count as a proof.

Other successful approaches included squaring $2x + 2y$ and using the circle equation to find that this square is equal to $4 + 8xy$, which will be maximised when xy is as great as possible. Exactly when this happens can be found for example by considering $(x - y)^2$, which is never negative, giving $xy \leq \frac{1}{2}(x^2 + y^2) = \frac{1}{2}$ and so the square of the perimeter is at most 8.

At least one student used the arithmetic mean–quadratic mean inequality very successfully to quickly solve the problem.

However, in many approaches students got to $P \leq 2\sqrt{2}$ but then didn't show that $P = 2\sqrt{2}$ can actually be achieved for some values of x and y: this is a vital step in showing that the maximum value really is $2\sqrt{2}$, and not something smaller.

Part (c) was found hard. Solving (c) does basically follow from the solution method for (b), but one has to notice that the problem in each quadrant can solve the big rectangle problem, or alternatively that one is scaling up from essentially finding the maximum sum of lengths of a right angled triangle of hypotenuse 1 to doing the same for a right angled triangle of hypotenuse 2. But a subtle point is that part (b) didn't ask for a proof that the *only way* to achieve a solution is with a square. In our official solution it was proved in (b) that the only way to maximise the perimeter is with a square, but many students when finishing a solution to (b) just showed that a square achieves the maximum possible perimeter – but maybe other rectangles could work too. So even for most students who solved (b) this was something new that was needed in (c). Marks were not awarded in (c) (or (b)) for assuming/claiming that the best possible shape is a square and then using that to work out the perimeter of that square.

Some students thought that, if a rectangle is inscribed in a circle, then it must be a square. Or possibly they inadvertently assumed this by labelling a right angle between the diagonals of the rectangle in part (c). As is always the case in questions involving geometrical figures, it is important to avoid unjustified assumptions about the shape being "special". You should always ask: 'Can I alter this special shape in some way while it still satisfies the conditions of the question?' Finally, some inexperienced students made algebraic mistakes in this question when squaring expressions or taking square roots of expressions. They may have therefore thought that they had solved the problem but, unfortunately for them, accurate algebra was needed here.

Question 5

Let n be a positive integer. Tracy writes a list of 10 whole numbers between 1 and n (inclusive). Each number in the list is either equal to, one less than, or one more than the number before it.

For example, when $n = 7$:

Her list could be 5, 5, 6, 7, 6, 6, 5, 5, 6, 6 or 4, 4, 3, 2, 1, 1, 1, 1, 1, 1.

Her list could *not* be 1, 3, 3, 4, 5, 5, 6, 7, 7, 7 or 5, 6, 7, 8, 7, 6, 5, 5, 5, 5.

(a) Suppose that $n = 3$. Stacey forms a list by copying Tracy's list, except that whenever Tracy writes a 1, Stacey writes a 3, and whenever Tracy writes a 3, Stacey writes a 1.

(i) Which lists could Tracy write that would cause her list to be the same as Stacey's?

(ii) Explain why Tracy can write as many lists that start 2, 2, 1 as start 2, 2, 3.

(3 marks)

(b) For which n between 1 and 10 (inclusive) is the number of lists that Tracy could write odd? (7 marks)

Solution

(a) (i) If Tracy writes a 3 at any point in her list, then Stacey will write a 1 at that point and so the lists will be different. Also, is Tracy writes a 1 at any point in her list, Stacey will write a 3 at that point and so the lists will be different. So, the only list that Tracy could possibly write that would cause Stacey to write the same list is the list where all ten entries are 2s. Indeed, this list does cause Stacey to have the same list as Tracy.

(ii) Note that whenever Tracy writes down a valid list of numbers, the list Stacey writes down is also a valid list of numbers. For any list L that Tracy can write, let $S(L)$ be the list that this causes Stacey to write down.

Suppose Tracy writes a list L that starts 2,2,1. Then $S(L)$ will start 2,2,3. We can get back to L from $S(L)$ by replacing all of the 1s in $S(L)$ with 3s and all of the 3s in $S(L)$ with 1s. In symbols, this says:

$$S(S(L)) = L.$$

So we can pair up each list L that Tracy could write beginning 2,2,1 with the list $S(L)$ beginning 2,2,3 that Stacey writes. Since this gives every list beginning 2,2,1 a unique partner beginning 2,2,3 (and vice versa), there must be the same number of lists beginning 2,2,1 as begin 2,2,3.

(b) Suppose for each n that whenever Tracy writes a list, Stacey copies Tracy's list except that whenever Tracy writes k, Stacey writes $n + 1 - k$. If Tracy writes a list L, write $S(L)$ again for the list that this causes Stacey to write.

Suppose Tracy writes a list which causes Stacey to write down the same list, i.e. $L = S(L)$. If an entry in Tracy's list is k, then we must have $k = n + 1 - k$, so $k = \frac{n+1}{2}$. Therefore, if

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n is odd there is one list where Stacey and Tracy write down the same list, namely

$$\frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}, \frac{n+1}{2}.$$

If n is even, then $\frac{n+1}{2}$ is not a whole number so there are no lists that cause Stacey and Tracy to write down the same list.

Now, if Tracy writes the list L and this causes Stacey to write the list $S(L)$, we can recover L from $S(L)$ by replacing each entry of $n+1-k$ with k, for $1 \leq k \leq n$. Since $n+1-(n+1-k) = k$, this is the same as doing Stacey's operation a second time, so

$$S(S(L)) = L.$$

So, all of the possible lists can be broken up into pairs of lists, where we pair the list L with its partner $S(L)$. If no lists are paired with themselves, this splits all of the possible lists into pairs, so there must be an even number of possible lists. Therefore, in the case n is even, there is an even number of possible lists. In the case n is odd, there is one list M that is paired with itself. So, the number of possible lists except for M is even, and therefore the total number of possible lists including M is one more than an even number. So, the total number of possible lists is odd whenever n is odd.

The possible values of n for which Stacey can write an odd number of lists are 1, 3, 5, 7 and 9.

Markers' comments

A few students were confused as to whether or not order mattered in forming the list, but seemed to get over this confusion after (a) (i) on realising that whether or not Tracy was allowed to write a certain list depended heavily on the order of its entries.

For part (a) (ii), many students tried to argue that there were the same number of options to follow a 1 as to follow a 3, and that the options to continue the list are identical after a 2. Such an argument can be made to work, but it implicitly uses the fact that there are the same number of lists beginning 2,2,3,3 as 2,2,1,1 which is about as hard to prove as the original problem. Students attempting this line of argument would have to explicitly note that there were the same number of lists beginning 2,2,1,2 as 2,2,3,2, beginning 2,2,1,1,2 as 2,2,3,3,2, beginning 2,2,1,1,1,2 as 2,2,3,3,3,2, and so on. However, many students noticed that replacing 1s with 3s and vice versa gave a symmetry that preserved the validity of the lists and successfully argued that they could pair lists beginning 2,2,1 with lists beginning 2,2,3.

Students who had taken a calculational approach to (a) (ii) tended to try to calculate the number of valid lists for each n between 1 and 10. Very few students had valid methods for doing this: the only correct method seen was recursive and required several pages of calculations. Attempts at direct counts often failed to take into account that after a 1 or an n there are 2 options for the next entry rather than 3, leading to the incorrect assertion that there are $n \cdot 3^9$ valid lists.

Some students took the idea of pairing from part (a) and tried to apply it in (b), correctly stating that the parity of the number of possible lists was the same as the parity of the number of lists paired with themselves. The hard part here is finding a way to change a list that both gives you a valid list when you apply the change, and can be undone by applying the change a second time. Most students attempting a pairing argument managed the latter by trying to swap entries for other entries, but often they just tried to swap 1 with n, which will not work in general for if $n \geq 4$, any list beginning 1,2 will be partnered with something beginning $n, 2$, which is not valid.

However, a good number of students spotted that reflecting in $\frac{n+1}{2}$ did the job, and went on to answer the question successfully.

An alternative way to pair the lists up is to consider reversing the order of the entries. This breaks the set of lists up into palindromic lists that are the same as their reversal, and non-palindromic lists, of which there are an even number. Hence, it suffices to calculate the parity of the number of palindromic lists of length 10 for each n. Notice that a palindromic list of length 10 is uniquely determined by its first 5 entries, and that any list of length 5 can be extended to a palindromic list of length 10 by concatenating the length 5 list with its own reversal, so the parity we are looking for is the parity of the number of valid lists of length 5. By doing the same thing again, this is the same as the parity of the number of palindromic lists of length 5, which is the same as the number of lists of length 3. Again, this is the same as the parity of the number of palindromic lists of length 3, which is the same as the parity of the number of lists of length 2, which is the same as the parity of the number of palindromic lists of length 2, which is the same as the number of lists of length 1. There are n lists of length 1, and therefore the number of lists of length 10 is odd if and only if n is odd.

A possible extension question looking at both the pairings at once is: what is the remainder when the number of possible lists is divided by 4, for each n? Can you find any other useful pairings?

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