United Kingdom
Mathematics Trust

Mathematical Olympiad for Girls

Tuesday 2nd October 2018

Organised by the United Kingdom Mathematics Trust

Instructions

1. Do not turn over the page until told to do so.
2. Time allowed: $2\frac{1}{2}$ hours.
3. Each question carries 10 marks. To gain full marks, your solution should be explained in full sentences. If your solution involves calculations, equations, tables, etc., explain where these come from and how you are using them. Explain how the steps of your solution link together, and give full proofs of assertions that you make. Answers alone will gain few marks (if any).

   Work in rough first, and then write up your best attempt at a clearly explained solution.
4. Partial marks may be awarded for good ideas, so try to hand in everything that documents your thinking on the problem — the more clearly written the better.

   However, one complete solution will gain more credit than several unfinished attempts.
5. Earlier questions tend to be easier. Some questions have multiple parts. Often earlier parts introduce results or ideas useful in solving later parts of the problem.
6. The use of rulers and compasses is allowed, but calculators and protractors are forbidden.
7. Start each question on a fresh sheet of paper. Write on one side of the paper only.

   On each sheet of working write the number of the question in the top left-hand corner and your name, initials and school in the top right-hand corner.
8. Complete the cover sheet provided and attach it to the front of your script, followed by your solutions in question number order.
9. Staple all the pages neatly together in the top left hand corner.
10. To accommodate candidates sitting in other time zones, please do not discuss the paper on the internet until 08:00 BST on Wednesday 3rd October.

Enquiries about the Mathematical Olympiad for Girls should be sent to:

MOG, UK Mathematics Trust, School of Mathematics Satellite,
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1. (a) Write down the full factorisation of the expression $a^2 - b^2$.

(i) Show that 359999 is not prime.

(ii) Show that 249919 is not prime.

Hint You can use your factorisation of $a^2 - b^2$ if you like. (5 marks)

(b) Write down the full factorisation of $a^2 + 2ab + b^2$.

(i) Show that 9006001 is not prime.

(ii) Show that 11449 is not prime.

(5 marks)

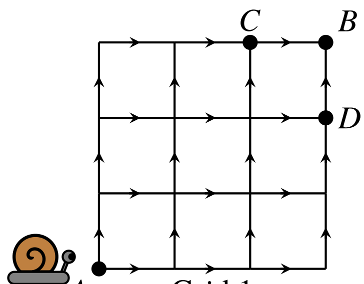
2. Triangle ABC is isosceles, with $AB = BC = 1$ and angle ABC equal to $120°$. A circle is tangent to the line AB at A and to the line BC at C.

What is the radius of the circle? (10 marks)

[*You should state clearly any geometrical facts or theorems you use in each step of your calculation. For example, if one of your steps calculates the size of an angle in a triangle you might justify that particular step with "because angles in a triangle add up to* 180 *degrees."*]

3. (a) Sheila the snail leaves a trail behind her as she moves along gridlines in Grid 1. She may only move in one direction along a gridline, indicated by arrows. Let b, c, d be the number of different trails Sheila could make while moving from A to B, C, D respectively.

Explain why $b = c + d$. (2 marks)



Grid 1

(b) Ghastly the ghost lives in a haunted mansion with 27 rooms arranged in a $3 \times 3 \times 3$ cube. He may pass unhindered between adjacent rooms, moving through the walls or ceilings. He wants to move from the room in the bottom left corner of the building to the room farthest away in the top right corner, passing through as few rooms as possible. Unfortunately, a trap has been placed in the room at the centre of the house and he must avoid it at all costs.

How many distinct paths through the house can he take? (8 marks)

Questions 4 and 5 are printed on the next page.

4. Each of 100 houses in a row are to be painted white or yellow. The residents are quite particular and request that no three neighbouring houses are all the same colour.

(a) Explain why no more than 67 houses can be painted yellow. (4 marks)

(b) In how many different ways may the houses be painted if exactly 67 are painted yellow? (6 marks)

5. Sophie lays out 9 coins in a 3×3 square grid, one in each cell, so that each coin is tail side up. A move consists of choosing a coin and turning over all coins which are adjacent to the chosen coin. For example if the centre coin is chosen then the four coins in cells above, below, left and right of it would be turned over.

(a) Sophie records the number of times she has chosen each coin in a 3×3 table. Explain how she can use this table to determine which way up every coin in the grid is at the end of a sequence of moves. (2 marks)

(b) Is it possible that after a sequence of moves all coins are tail side down? (4 marks)

(c) If instead Sophie lays out 16 coins in the cells of a 4×4 grid, so that each coin is tail side up, is it possible that after a sequence of moves all coins are tail side down? (4 marks)

[*In parts* (*b*) *and* (*c*), *if you think that it is possible, you should specify a sequence of moves, after which all coins are tail side down. If you think it is not possible, you should give a proof to show that it can't be done, no matter which sequence of moves Sophie chooses to do.*]