The 6^{th} Romanian Master of Mathematics Competition

Day 1: Friday, March 1, 2013, Bucharest

Language: English

Problem 1. For a positive integer a, define a sequence of integers $x_1, x_2, \ldots$ by letting $x_1 = a$ and $x_{n+1} = 2x_n + 1$ for $n \geq 1$. Let $y_n = 2^{x_n} - 1$. Determine the largest possible k such that, for some positive integer a, the numbers $y_1, \ldots, y_k$ are all prime.

Problem 2. Does there exist a pair (g, h) of functions $g, h \colon \mathbb{R} \to \mathbb{R}$ such that the only function $f \colon \mathbb{R} \to \mathbb{R}$ satisfying $f(g(x)) = g(f(x))$ and $f(h(x)) = h(f(x))$ for all $x \in \mathbb{R}$ is the identity function $f(x) \equiv x$?

Problem 3. Let $ABCD$ be a quadrilateral inscribed in a circle ω. The lines AB and CD meet at P, the lines AD and BC meet at Q, and the diagonals AC and BD meet at R. Let M be the midpoint of the segment PQ, and let K be the common point of the segment MR and the circle ω. Prove that the circumcircle of the triangle KPQ and ω are tangent to one another.

Each of the three problems is worth 7 points.
Time allowed $4\frac{1}{2}$ hours.