From Classroom to Contest

UKMT
2016 IMO Celebration

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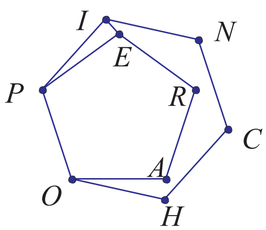
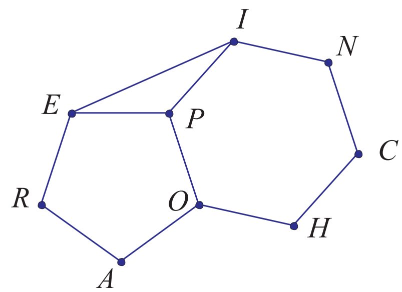
1 Connecting (regular) polygons

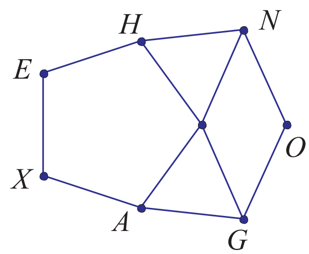
1. (Based on MPG 2016) In convex equilateral heptagon $HEXAGON$, $\angle H = \angle A = 168^\circ$ and $\angle E = \angle X = 108^\circ$. Find the respective degree measures of $\angle G, \angle O, \angle N$.

2. Let $CHOPIN$ be a regular hexagon, and let $OPERA$ be a regular pentagon. Find all possible values of measure of $\angle PIE$.





Solution: The values are $84°$ and $24°$.


H
N
E
O
X
A
G

2 How an IMO problem was written from an Algebra 1 class

3. Find three distinct integers values of n such that $1 + 2^4 + 2^n$ is a perfect square.

Note that $1+2^4+2^6=(1+2^3)^2$ and $1+2^3+2^4=(1+2^2)^2$. Two of the possible values of n could be $n=6$ and $n=3$.

We are simply playing with the algebra fact $1 + 2^{n+1} + 2^{2n} = (1 + 2^n)^2$.

It is easy to check that $1 + 2^4 + 2^n = 17 + 2^n = 49 = 7^2$ when $n = 5$. Why does this work? What is the algebra behind this fact?

We notice that $1 + 2^4 + 2^5 = 1 - 2^4 + 2^5 + 2^5 = 1 - 2^4 + 2^6 = (2^3 - 1)^2$.

Are there any more possible values of n?

A quick computation checks $1 + 2^4 + 2^9 = 529 = 23^2$.

Interestingly, we note that $1 + 2^4 + 2^9$ is in the form of $1 + 2^n + 2^{2n+1}$ rather than $1 + 2^{n+1} + 2^{2n} = (1 + 2^n)^2$, which is the algebraic identity we have applied before.

4. (IMO 2006, by Zuming Feng) Determine all pairs (x, y) of integers such that

$$1 + 2^x + 2^{2x+1} = y^2.$$

Multiplying both sides of the given equation by 8 gives

$$8 + 2^{x+3} + 2^{2x+4} = 8y^2 \quad \text{or} \quad 8y^2 - \left(2^{x+2} + 1\right)^2 = 7,$$

which relates to the Pell's equation $2A^2 - B^2 = 7$.

It is easy to check that there is no solution for $x = 1, 2,$ and 3. We assume that (x, y) is a solution with $x \geq 5$ and $y > 0$. Note that

$$\begin{cases} 1 + 2^x + 2^{2x+1} = y^2 \\ 1 + 2^{x+1} + 2^{2x} = (1 + 2^x)^2. \end{cases}$$

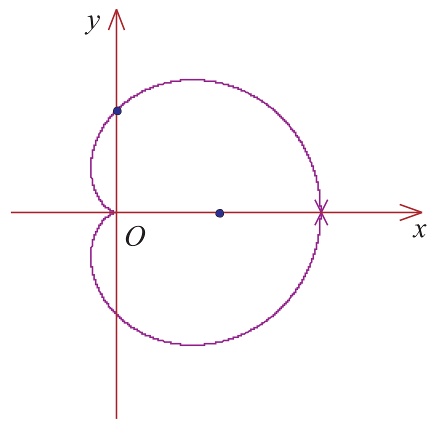
Subtracting the two equations gives

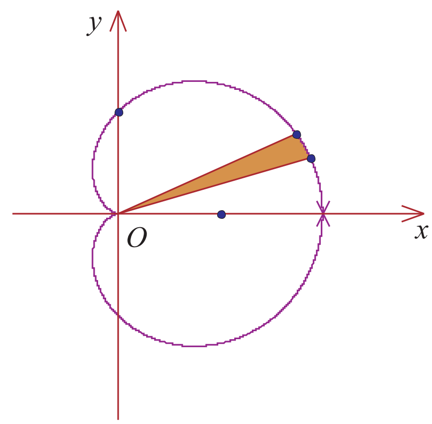
$$[y - (1 + 2^x)][y + (1 + 2^x)] = 2^{2x} - 2^x = 2^x(2^x - 1).$$

3 Polar coordinates

5. (RMM 2013, by Nikolai Beluhov from Bulgaria) Let $\mathcal{P}$ and $\mathcal{Q}$ be two convex quadrilateral regions in the plane which have a common point O (regions contain their boundary). Suppose that for every line ℓ through O the segment of intersection of ℓ and $\mathcal{P}$ is longer than the segment of intersection of ℓ and $\mathcal{Q}$. Is it possible that the ratio of the area of $\mathcal{Q}$ to the area of $\mathcal{P}$ is greater than 1.9?

6. The diagram shows the *cardioid* described by the polar equation $r = 1 + \cos\theta$. Use integration to find the area of the region enclosed by this curve.


y
O
x



y
O
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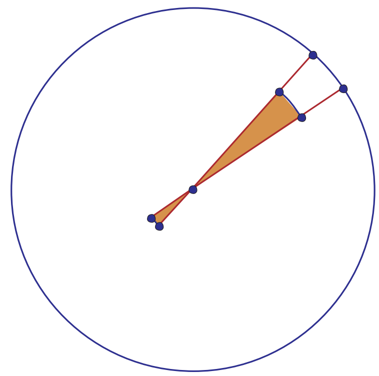
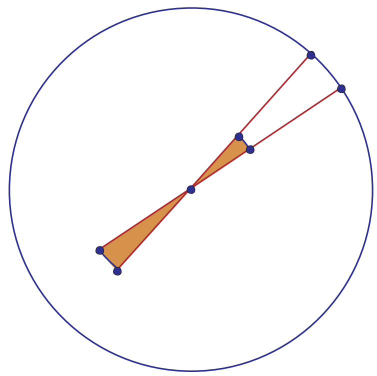
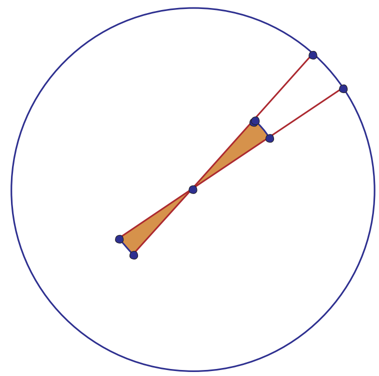
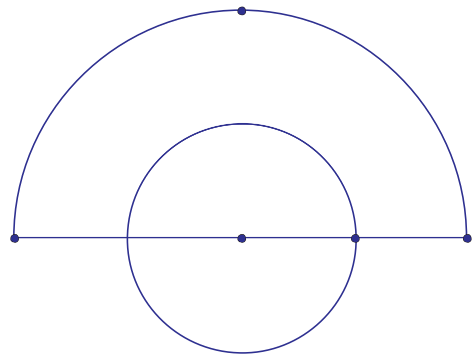
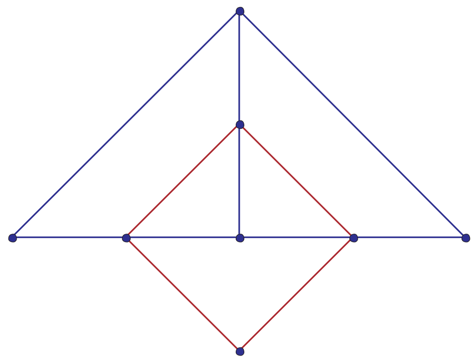
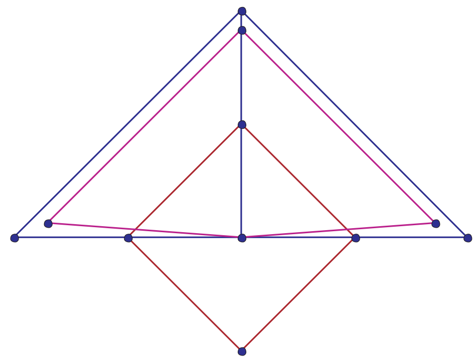






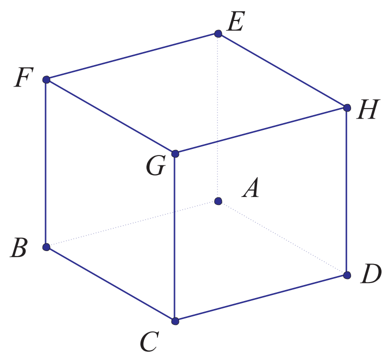




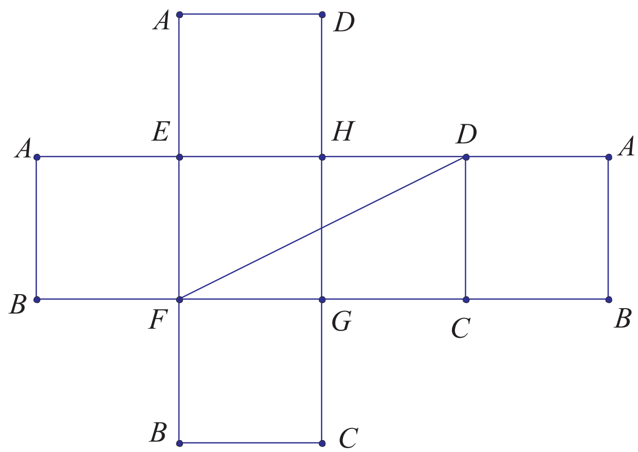


4 Box and bugs

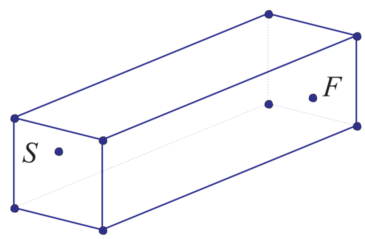
7. In the figure shown below, an ant is positioned at F, one of the eight vertices of a solid cube. It needs to crawl to vertex D, which is the furthest vertex from F, as fast as possible. Find one of the shortest routes. How many are there?

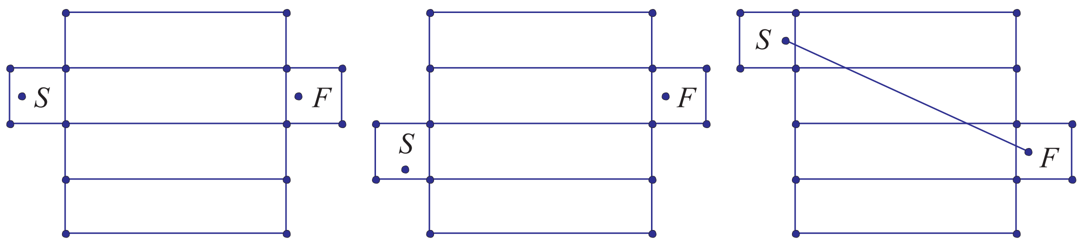


Folding open the box, for instance, hinging it along edge GH as shown below, will do the job. We only need to find the shortest path on a plane, which is easy to do. There are 6 such routes: heading for the midpoint of a nonadjacent edge on each face containing F.



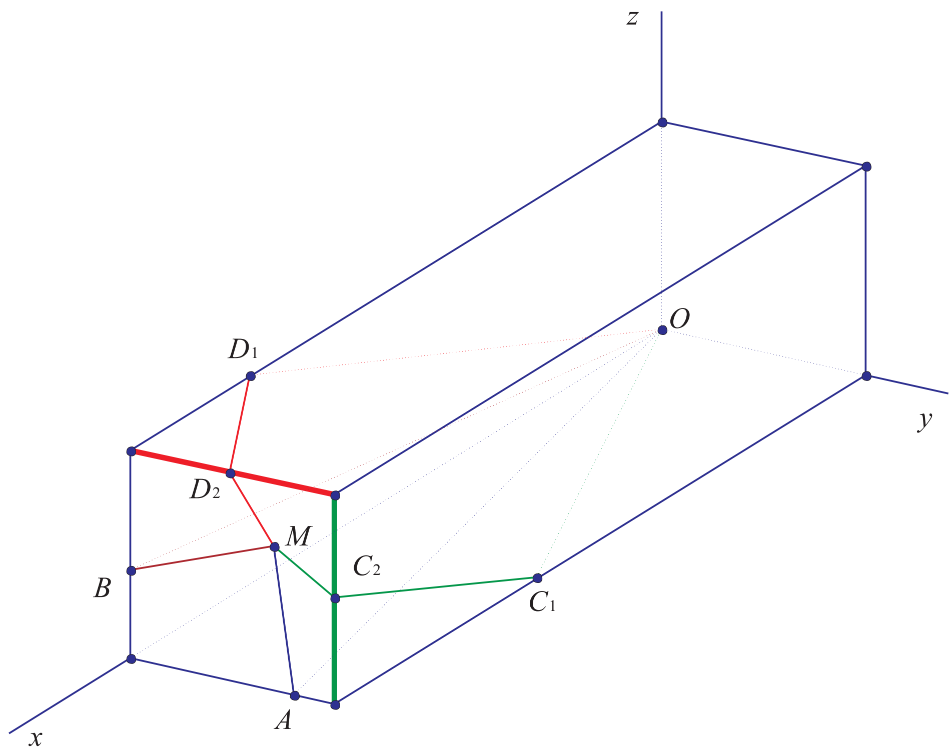
8. In the diagram shown below, a spider lived in a room that measured 30 feet long by 12 feet wide by 12 feet high. One day, the spider spied an incapacitated fly across the room, and of course wanted to crawl to it as quickly as possible. The spider was on an end wall, one foot from the ceiling and six feet from each of the long walls. The fly was stuck one foot from the floor on the opposite wall, also midway between the two long walls. Knowing some geometry, the spider cleverly took the shortest possible route to the fly and ate it for lunch. How far did the spider crawl?



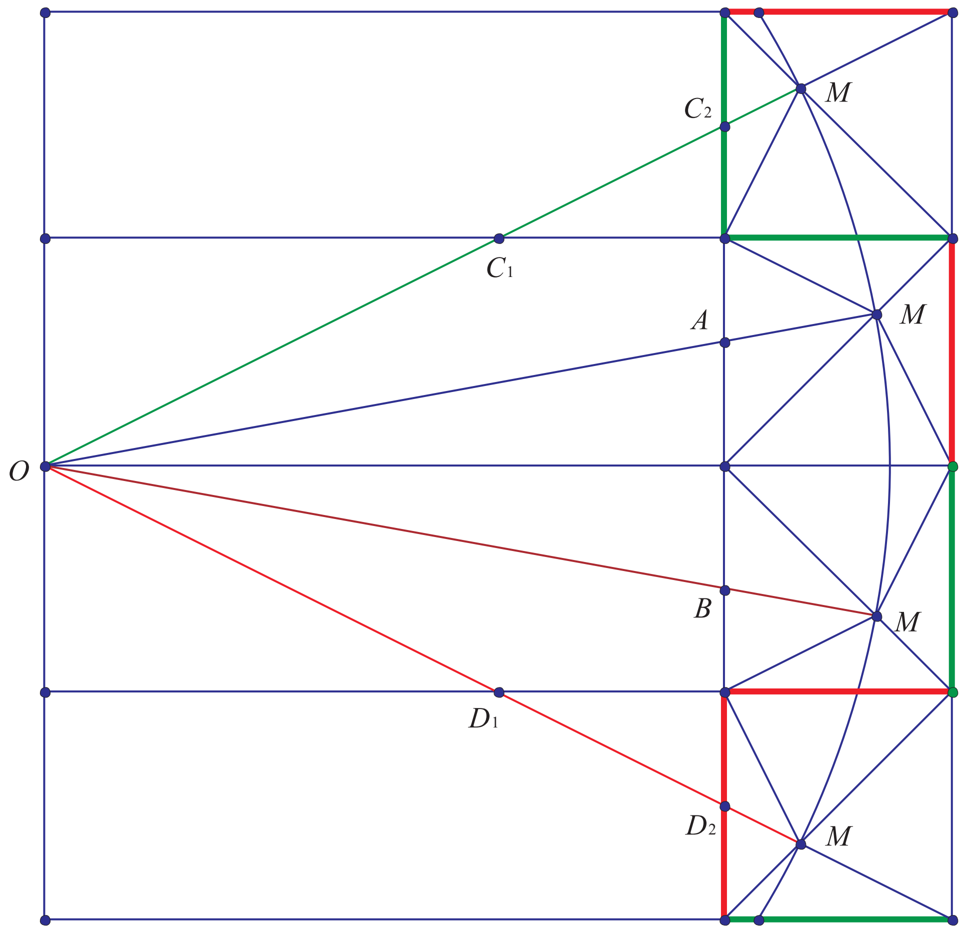


(In 3-D, there are two such paths, symmetrically to each other across the plane passing through the ant and the spider perpendicular to the floor. Each of these two paths passes is composed of 5 segments lying on the 5 different faces of the room.)

9. (AIME 2009 packet, By Richard Parris) Let C be a corner of a $3 \times 1 \times 1$ rectangular solid $\mathcal{R}$, and let P be the non-interior point of $\mathcal{R}$ whose distance d from C, when measured along the surface of $\mathcal{R}$, is maximal. Given that $d^2 = m/n$, where m and n are relatively prime positive integers, find $m + n$.



The solid $\mathcal{R}$ can be described by the inequalities $0 \leq x \leq 3$, $0 \leq y \leq 1$, and $0 \leq z \leq 1$. Let $O = (0, 0, 0)$. It is evident that M is somewhere on the square face S that is not adjacent to O, and thus $M = (3, y, z)$. The distance d from O to M is the length of the shortest piecewise-linear path from O to M. As the diagram shows, there are four candidates for the shortest path. namely, $O - A - M$, $O - B - M$, $O - C_1 - C_2 - M$, and $O - D_1 - D_2$.



These paths can be made linear by *unfoldingh* the surface of the box. In this way, it is seen that $d(y,z)^2$ is the minimum of

$$
\begin{aligned}
f_1(y,z) &= (3+z)^2 + y^2, \\
f_2(y,z) &= (3+y)^2 + z^2, \\
f_4(y,z) &= (4-z)^2 + (1+y)^2, \\
f_4(y,z) &= (4-y)^2 + (1+z)^2.
\end{aligned}
$$

Pairwise comparisons shows that

$$\begin{aligned}
f_1(y,z) \leq f_2(y,z) &\quad \text{when} \quad z \leq y, \\
f_1(y,z) \leq f_3(y,z) &\quad \text{when} \quad 7z \leq 4+y, \\
f_1(y,z) \leq f_4(y,z) &\quad \text{when} \quad z+2y \leq 2, \\
f_2(y,z) \leq f_3(y,z) &\quad \text{when} \quad y+2z \leq 2, \\
f_2(y,z) \leq f_4(y,z) &\quad \text{when} \quad 7y \leq 4+z, \\
f_3(y,z) \leq f_4(y,z) &\quad \text{when} \quad y \leq z.
\end{aligned}$$

It is not evident that the $M = \left(3, \frac{2}{3}, \frac{2}{3}\right)$ (the intersection of these three regions) and $d^2 = \frac{125}{9}$.

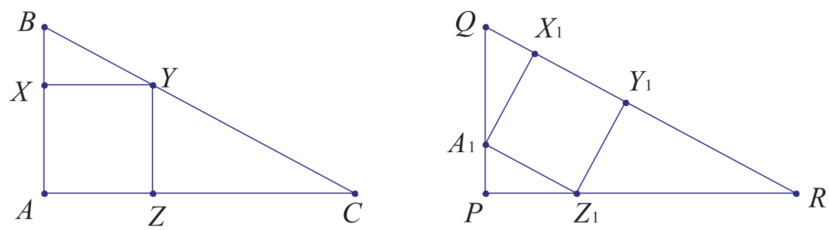
What if the dimension of the box is $1 \times a \times b$? If you are interested in this, please visit

http://www.exeter.edu/documents/Math6All.pdf

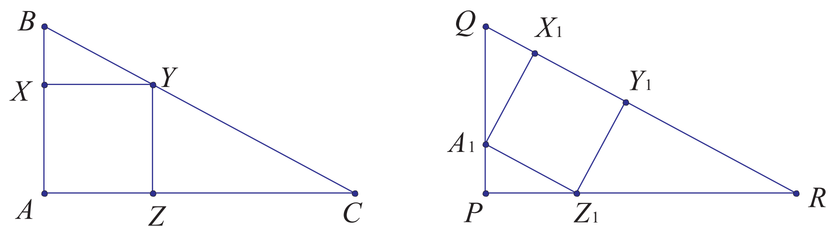
to see the problems on pages 65 and 66.

5 Square inside a triangle

10. (AIME 1987) Squares S_1 and S_2 are inscribed in the two congruent right triangle ABC and PQR, as shown below. Find $AB + AC$ if $[AXYZ] = 441$ and $[A_1X_1Y_1Z_1] = 440$.



11. How large a square can be put inside a right triangle whose legs are 5 cm and 12 cm?



Set $XY = s$. By similar triangles BXY and BAC, we have $BY/XY = BC/AC$ or $BY = sa/b$. Likewise, $CY = sa/c$. Because $BY + CY = a$, we have

$$\frac{sa}{b} + \frac{sa}{c} = a \quad \text{or} \quad s = \frac{bc}{b+c}.$$

Set $A_1X_1 = t$. By similar triangles A_1QX_1 and QPR, we have $QX_1/A_1X_1 = PQ/PR = c/b$ or $QX_1 = tc/b$. Likewise, $RY_1 = tb/c$. Because $QX_1 + X_1Y_1 + Y_1R = QR = a$, we have

$$\frac{tb}{c} + t + \frac{tc}{b} = a \quad \text{or} \quad t = \frac{abc}{b^2 + c^2 + bc}.$$

For an arbitrary right triangle, will the first method always result in a bigger inscribed square?

It remains to show that

$$\frac{abc}{b^2 + c^2 + bc} = t < s = \frac{bc}{b + c},$$

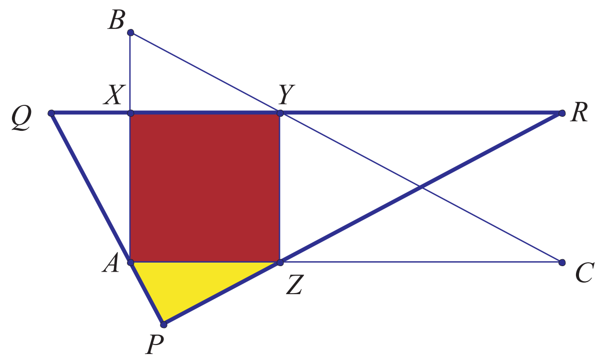
or $a(b + c) < (a^2 + bc)$. The inequality can easily be established by squaring both sides and expanding as

$$(a^2 + bc)^2 = a^4 + 2a^2bc + b^2c^2$$

and

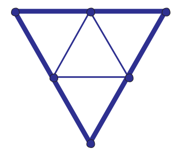
$$a^2(b + c)^2 = a^2(b^2 + c^2 + 2bc) = a^2(a^2 + 2bc) = a^4 + 2a^2b.$$

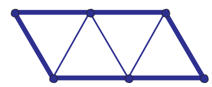
We inscribe the same square ($AXYZ$) into two similar right triangles. The second method requires a bigger triangle (triangle PQR), and so the first method (triangle ABC) is better. (Note that triangles YZR and YZC are congruent to each other, and so do triangles BXY and AXQ.)

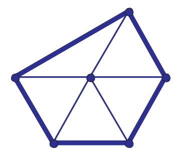


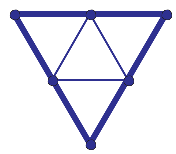
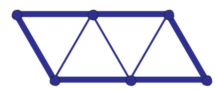
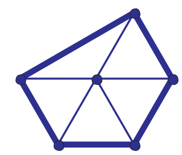
6 Gems from contests run by students

12. (EMCC 2012) Farmer Chong Gu glues together 4 equilateral triangles of side length 1 such that their edges coincide. He then drives in a stake at each vertex of the original triangles and puts a rubber band around all the stakes. Find the minimum possible length of the rubber band.

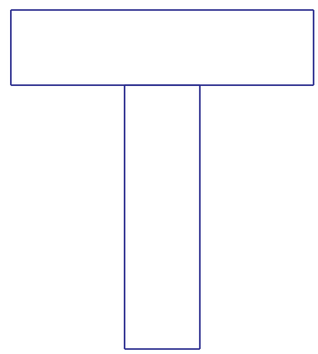


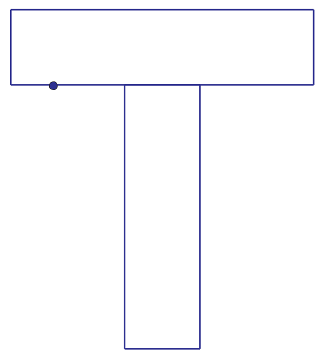


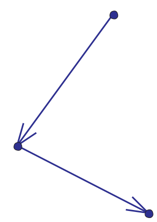


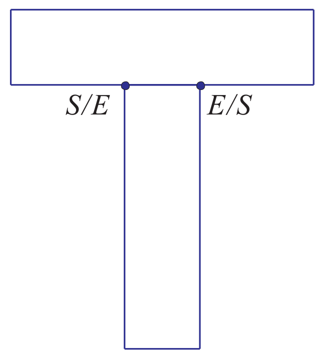
  

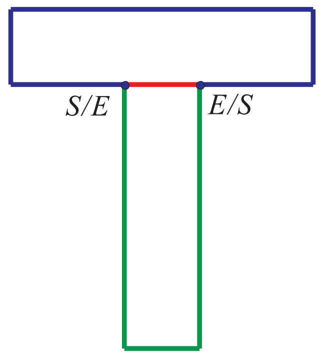
13. (EMCC 2012) In how many ways can Alex draw the diagram below without lifting his pencil or retracing a line? (Two drawings are different if the order in which he draws the edges is different, or the direction in which he draws an edge is different).



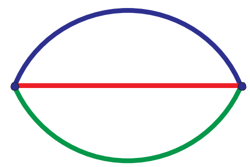


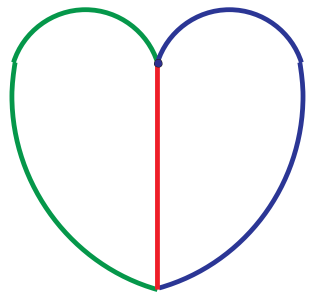



S/E
E/S

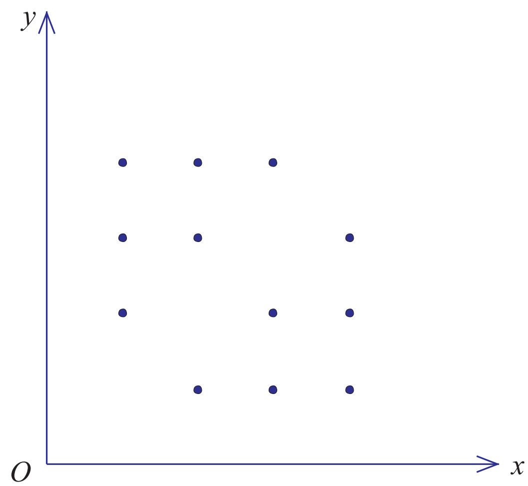


S/E
E/S

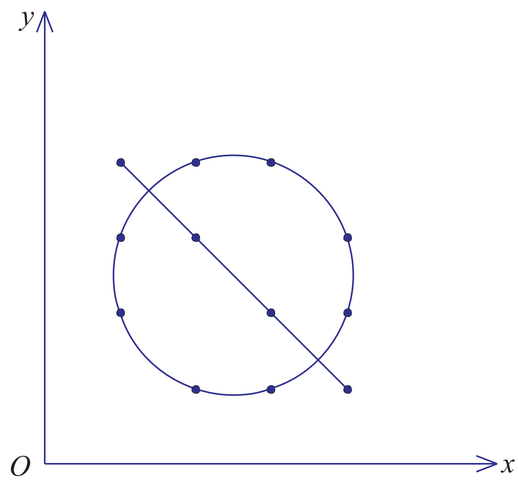




14. (EMCC 2014, by Zhuoqun Alex Song) We say a polynomial P in x and y is n-good if $P(x, y) = 0$ for all integers x and y, with $x \neq y$, between 1 and n, inclusive. Determine the minimal degree of a nonzero 4-good polynomial.



y
O
x


y
O
x

15. (ELMO 2006, by Baohua Zhan) Numbers from 1 to n^2 is put into an $n \times n$ square array with no repetitions. A path is a series of movements from the bottom left to the top right corner of the square, moving only up or to the right (all paths cover exactly $2n-1$ squares). The weight of a path is the sum of all numbers covered by the path. For any numbered square we can find a path with largest weight and a path with the smallest weight. Among all possible ways to number a square, what is the smallest possible difference between the weights of these two paths?

7 Is this geometry?

16. In the city of Exercise, the capital of the Fat Republic, 2016 families who are unhappy with their current apartment are part of a project to exchange apartments. Each family is allowed to exchange apartments with another family once each day; only pairwise apartment exchanges are allowed. (Multiple pairs of families can exchange apartments in any given day.) It is known that there is a way of pairing families with apartments so that each family is in an apartment that it likes. What is the minimum number of days needed to guarantee that the families can switch to this arrangement, regardless of what that arrangement is?