*Wednesday, July 24, 2013*

**Problem 4.** Let $ABC$ be an acute-angled triangle with orthocentre $H$, and let $W$ be a point on the side $BC$, lying strictly between $B$ and $C$. The points $M$ and $N$ are the feet of the altitudes from $B$ and $C$, respectively. Denote by $\omega_1$ the circumcircle of $BWN$, and let $X$ be the point on $\omega_1$ such that $WX$ is a diameter of $\omega_1$. Analogously, denote by $\omega_2$ the circumcircle of $CWM$, and let $Y$ be the point on $\omega_2$ such that $WY$ is a diameter of $\omega_2$. Prove that $X$, $Y$ and $H$ are collinear.

**Problem 5.** Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let $f\colon \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:

(i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$;

(ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x) + f(y)$;

(iii) there exists a rational number $a > 1$ such that $f(a) = a$.

Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

**Problem 6.** Let $n \geq 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, \ldots, n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called *beautiful* if, for any four labels $a < b < c < d$ with $a+d = b+c$, the chord joining the points labelled $a$ and $d$ does not intersect the chord joining the points labelled $b$ and $c$. Let $M$ be the number of beautiful labellings, and let $N$ be the number of ordered pairs $(x, y)$ of positive integers such that $x + y \leq n$ and $\gcd(x, y) = 1$. Prove that

$$M = N + 1.$$

*Language: English*

*Time: 4 hours and 30 minutes*
*Each problem is worth 7 points*