Tuesday, July 23, 2013

Problem 1. Prove that for any pair of positive integers k and n, there exist k positive integers $m_1, m_2, \ldots, m_k$ (not necessarily different) such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right) \cdots \left(1 + \frac{1}{m_k}\right).$$

Problem 2. A configuration of 4027 points in the plane is called *Colombian* if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is *good* for a Colombian configuration if the following two conditions are satisfied:

- no line passes through any point of the configuration;
- no region contains points of both colours.

Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

Problem 3. Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

The excircle *of triangle ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C. The excircles opposite B and C are similarly defined.*

Language: English

Time: 4 hours and 30 minutes
Each problem is worth 7 points