First Selection Test: Exam 2

IMO camp, Trinity College Cambridge

7-iv-2008

Problem 1 Let n be a positive integer, and let x and y be positive real numbers such that $x^n + y^n = 1$. Prove that

$$\left(\sum_{k=1}^{n} \frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n} \frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{1}{(1-x)(1-y)}.$$

Problem 2 Find all positive integers n for which the numbers in the set

$$S = \{1, 2, \ldots, n\}$$

can each be coloured either red or blue such that the following conditions are satisfied: there are exactly 2007 ordered triples (x, y, z) of elements of S such that

(i) x, y, z have the same colour and

(ii) $x + y + z$ is divisible by n.

Problem 3 Let ABC be a fixed triangle, and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB respectively. Let P be a variable point on the circumcircle Σ of ABC. Let the lines PA_1, PB_1, PC_1 meet Σ again at A', B', C' respectively. Assume that the points A, B, C, A', B', C' are distinct and so the lines AA', BB', CC' form a triangle. Prove that the area of this triangle does not depend on P.

Time allowed: 4 hours 30 minutes