NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

FURTHER INTERNATIONAL SELECTION TEST 1990

Wednesday, 7th March 1990

Time allowed: Three-and-a-half hours

- *Arrange your answers in order, with your name on each page.*
- *Complete the proforma provided and attach it to the front of your script.*

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1. Prove that if the polynomial

$$p(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n ,$$

whose coefficients are integers, takes the value 1990 for four distinct integer values of x, then it cannot take the value 1997 for any integer value of x.

2. The *integer part* $[x]$ of a number x is the greatest integer which is not greater than x. The *fractional part* $f(x)$ is defined by $f(x) = x - [x]$.
Find a positive number x such that

$$f(x) + f\left(\frac{1}{x}\right) = 1 .$$

Are there any *rational* solutions?

3. State the cosine rule for a triangle.
Prove that for arbitrary positive real numbers a, b, c,

$$\sqrt{a^2 + b^2 - ab} + \sqrt{b^2 + c^2 - bc} \geq \sqrt{a^2 + c^2 + ac} .$$

4. Let d denote the length of the smallest diagonal of all rectangles inscribed in a triangle T. (By inscribed we mean that all vertices of the rectangle lie on the boundary of T).
Determine the maximum value of d^2/area(T) taken over all triangles.

5. I is the centre of the circle inscribed to triangle ABC; J is the centre of the escribed circle which touches AB and AC produced beyond B and C respectively.
Prove that

$$AI.AJ = AB.AC$$

and that

$$AI.BJ.CJ = AJ.BI.CI.$$
