NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test.

Friday, 14th March 1986

Time allowed $3\frac{1}{2}$ hours.

Write on one side of the paper only. Start each question on a fresh sheet of paper. Arrange your answers in order. Put your full name, age (in years and months) and school on the top sheet of your answers. On each other sheet put your name and initials.

1. Plane rectangular Cartesian axes are given with equal unit length along each axis. A rational point is defined as a point both of whose coordinates are rational numbers.

   A, B, A', B' are four distinct rational points; A and B are on the x-axis. Prove that, unless $\vec{AB} = \vec{A'B'}$, there exists just one point P such that the triangles PAB, $PA'B'$ are directly similar, i.e. each can be obtained from the other by enlargement (dilatation) and rotation about P. Prove also that P is a rational point.

2. Find, with proof, the greatest value of
$$x^2y + y^2z + z^2x$$
   where x,y,z are real numbers satisfying the conditions
$$x + y + z = 0, \quad x^2 + y^2 + z^2 = 6.$$

3. $P_1, P_2, \ldots, P_n$ are n distinct subsets of $\{1,2,\ldots,n\}$ each having two elements. P_i and P_j $(i \neq j)$ have an element in common if and only if $\{i,j\}$ is one of the subsets P_k. Prove that each of $1,2,\ldots,n$ belongs to exactly two of the P_k.

4. Show that if m,n are positive integers with $m \leqslant n$ then the product $\binom{n}{m}\binom{n}{m-1}$ of binomial coefficients is divisible by n. Find with proof the smallest positive integer k such that the product $k\binom{n}{m}\binom{n}{m-1}\binom{n}{m-2}$ is divisible by n^2 for all integers m,n with $2 \leqslant m \leqslant n$. [$k$ is to be independent of m,n.]
   For this k and given n determine, with proof, the greatest common divisor of the integers $\frac{k}{n^2}\binom{n}{m}\binom{n}{m-1}\binom{n}{m-2}$, $2 \leqslant m \leqslant n$.

5. C_1 and C_2 are two circles; A_1, A_2 are fixed points on C_1, C_2 respectively. A_1P_1, A_2P_2 are parallel chords of C_1, C_2. Find the locus of the midpoint of P_1P_2.