NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

Further International Selection Test, 1976

May 5th, 3½ hours

1. Through a point P in the interior of a fixed triangle ABC lines PL, PM, PN are drawn parallel to the medians through A,B,C respectively to meet BC, CA, AB at L,M,N respectively. Prove -

$$\frac{BL}{BC} + \frac{CM}{CA} + \frac{AN}{AB}$$

is constant (independent of P).

2. The real number t is a root of the equation

$$x^n + a_2x^{n-2} + a_3x^{n-3} + \ldots\ldots + a_n = 0, \quad (n \geqslant 2),$$

where the coefficients are real and satisfy $-1 \leqslant a_r \leqslant 1$, $(2 \leqslant r \leqslant n)$.

Prove that $$-\tfrac{1}{2}(1 + \sqrt{5}) \leqslant t \leqslant \tfrac{1}{2}(1 + \sqrt{5}).$$

3. Prove that the equation $x^2 - 3y^2 + 5z^2 - 7t^2 = 0$ has no solutions in integers x,y,z,t other than x=y=z=t=0.

Prove that $x^2-3y^2-5z^2+7t^2 = 0$ has infinitely many solutions in positive integers x,y,z,t in no two of which the ratio x:y:z:t is the same.

4. Prove that it is not possible to find positive integers p and q with the property that

$$\left|\frac{p}{q} - \sqrt{7}\right| \leqslant \frac{2}{11q^2}$$

5. A 'figure-of-eight' curve, S, consists of two touching circles of equal radii. Show that a pair of two distinct congruent hexagons (not necessarily convex) exists with the following properties:

(a) All the vertices of the hexagons lie on S.

(b) Neither hexagon has all its vertices on one circle.

(c) Neither hexagon can be obtained from the other by a single translation, a single rotation or a single reflection.