United Kingdom Mathematics Trust

British Mathematical Olympiad

Round 2 : Thursday 25 January 2018

Time allowed *Three and a half hours.*

Each question is worth 10 marks.

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*

  Rough work should *be handed in, but should be clearly marked.*
- *One or two* complete *solutions will gain far more credit than partial attempts at all four problems.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Staple all the pages neatly together in the top* left *hand corner, with questions* 1, 2, 3, 4 *in order, and the cover sheet at the front.*
- *To accommodate candidates sitting in other time zones, please do not discuss any aspect of the paper on the internet until 8am GMT on Friday 26 January. Candidates sitting the paper in time zones more than 3 hours ahead of GMT must sit the paper on Friday 26 January (as defined locally).*

In early March, twenty students eligible to represent the UK at the International Mathematical Olympiad will be invited to attend the training session to be held at Trinity College, Cambridge (4–9 April 2018). At the training session, students sit a pair of IMO-style papers and eight students will be selected for further training and selection examinations. The UK Team of six for this year's IMO (to be held in Cluj–Napoca, Romania 3–14 July 2018) will then be chosen.

Do not turn over until **told to do so.**

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2017/18 British Mathematical Olympiad
Round 2

1. Consider triangle ABC. The midpoint of AC is M. The circle tangent to BC at B and passing through M meets the line AB again at P. Prove that $AB \times BP = 2BM^2$.

2. There are n places set for tea around a circular table, and every place has a small cake on a plate. Alice arrives first, sits at the table, and eats her cake (but it isn't very nice). Next the Mad Hatter arrives, and tells Alice that she will have a lonely tea party, and that she must keep on changing her seat, and each time she must eat the cake in front of her (if it has not yet been eaten). In fact the Mad Hatter is very bossy, and tells Alice that, for $i = 1, 2, \ldots, n-1$, when she moves for the i-th time, she must move a_i places and he hands Alice the list of instructions $a_1, a_2, \ldots, a_{n-1}$. Alice does not like the cakes, and she is free to choose, at every stage, whether to move clockwise or anticlockwise. For which values of n can the Mad Hatter force Alice to eat all the cakes?

3. It is well known that, for each positive integer n,
$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$
and so is a square. Determine whether or not there is a positive integer m such that
$$(m+1)^3 + (m+2)^3 + \cdots + (2m)^3$$
is a square.

4. Let f be a function defined on the real numbers and taking real values. We say that f is *absorbing* if $f(x) \leq f(y)$ whenever $x \leq y$ and $f^{2018}(z)$ is an integer for all real numbers z.

   a) Does there exist an absorbing function f such that $f(x)$ is an integer for only finitely many values of x?

   b) Does there exist an absorbing function f and an increasing sequence of real numbers $a_1 < a_2 < a_3 < \ldots$ such that $f(x)$ is an integer only if $x = a_i$ for some i?

Note that if k is a positive integer and f is a function, then f^k denotes the composition of k copies of f. For example $f^3(t) = f(f(f(t)))$ for all real numbers t.