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BRITISH MATHEMATICAL OLYMPIAD ROUND 1 2021

Teachers are encouraged to distribute copies of this report to candidates.

Markers' report

The 2021 paper

Olympiad marking

Both candidates and their teachers will find it helpful to know something of the general principles involved in marking Olympiad-type papers. These preliminary paragraphs therefore serve as an exposition of the 'philosophy' which has guided both the setting and marking of all such papers at all age levels, both nationally and internationally.

What we are looking for are full solutions to problems. This involves identifying a suitable strategy, explaining why your strategy solves the problem, and then carrying it out to produce an answer or prove the required result. In marking each question, we look at the solution synoptically and decide whether the candidate has a viable overall strategy or not. An answer which is essentially a solution will be awarded near maximum credit, with marks deducted for errors of calculation, flaws in logic, omission of cases or technical faults. On the other hand, an answer which does not present a complete argument is marked on a '0 plus' basis; up to 4 marks might be awarded for particular cases or insights. If a problem has two distinct logical parts, these are sometimes marked separately and the scores added, but one part is generally considered the crux of the problem. For example, in Q2 we need to show (i) that Arun and Disha played at least 30 games and (ii) that they might have played exactly 30. Here (i) requires more sophistication, and carries 7 of the 10 marks available. In general the logical structure of the mark scheme aims to reflect the logical structure of the problem while rewarding correct arguments more generously than correct calculations.

This approach is therefore rather different from what happens in public examinations such as GCSE, AS and A level, where credit is given for the ability to carry out individual techniques regardless of how these techniques fit into a protracted argument. It is therefore vital that candidates taking Olympiad papers realise the importance of the comment in the rubric about trying to finish whole questions rather than attempting lots of disconnected parts.

General comments

Candidates found this a demanding BMO1 paper, though the vast majority still engaged well with one or two of the problems. Even very strong candidates found there was a lot to do in the time available, with many writing a great deal. It was encouraging to see candidates taking the requirement to provided full written solutions seriously, but it is worth pointing out that, while proofs generally require English sentences as well as mathematical symbols, a fairly concise style often adds clarity.

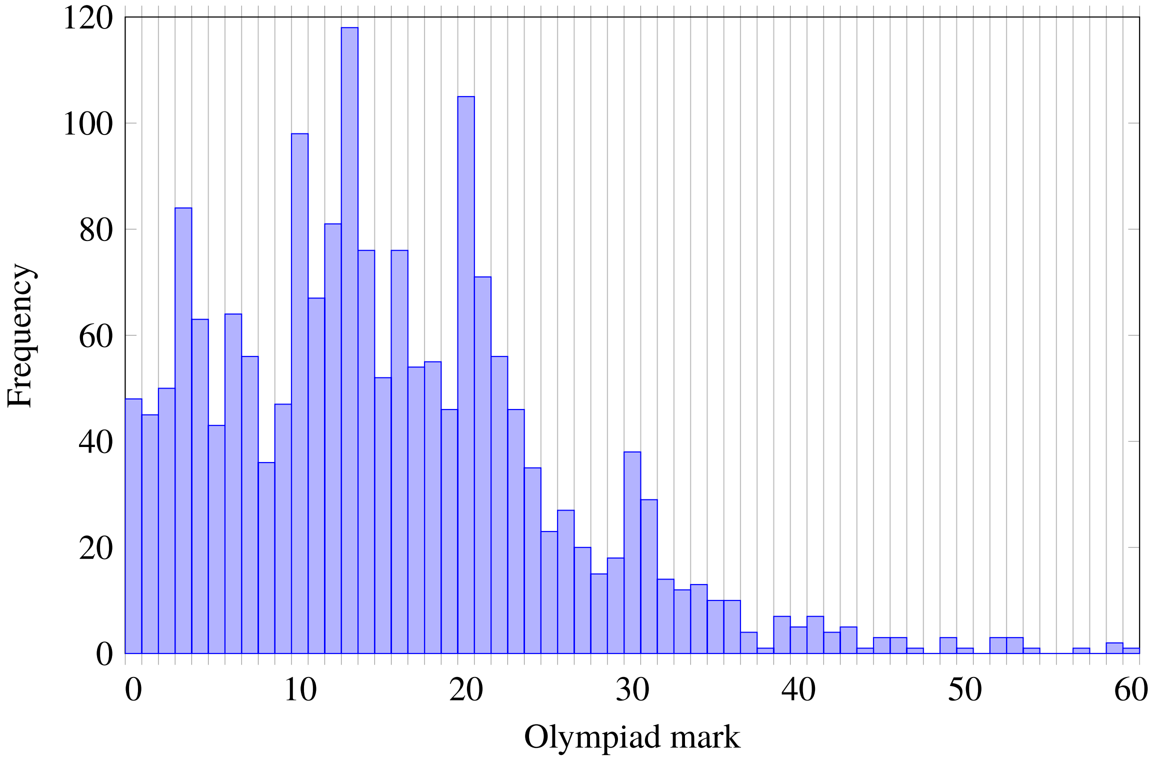
Markers were pleased to see that, while five of this year's problems had numerical answers, almost all candidates realised that simply providing the 'correct number' was not what was required.

Since only one candidate scored full marks, we hope that others, particularly those not yet in their final school year, will forgive the repetition of some (hopefully) familiar advice:

- *Check your work*: many of the candidates could have scored more highly on question 1 if they had spent perhaps five minutes checking that their proposed numbers actually worked as intended.
- *Read the question*: many candidates simply missed the crucial phrase 'in some order' in question 2.
- *Try small examples*: many candidates did not test their ideas in question 3 on smaller numbers than 2021. Systematically considering the number of ways to make piles of 1, 2, 3,. . . grams of gold leads to a clear pattern fairly swiftly.

The 2021 British Mathematical Olympiad Round 1 attracted 1857 entries. The scripts were marked digitally from the 5th to the 15th of December by a team of Eszter Backhausz, Agnijo Banerjee, Sam Bealing, Emily Beatty, Jonathan Beckett, Natalie Behague, James Bell, Robin Bhattacharyya, Maya Brock, Magdalena Burrows, Andrea Chlebikova, James Cranch, Stephen Darby, Joe Devine, Ceri Fiddes, Richard Freeland, Carol Gainlall, Amit Goyal, Ben Handley, Stuart Haring, Adrian Hemery, Ina Hughes, Ian Jackson, Shavindra Jayasekera, Vesna Kadelburg, Adam Kelly, Jeremy King, Patricia King, David Knipe, Gordon Lessells, Rhys Lewis, Warren Li, Samuel Liew, Linus Luu, Nick MacKinnon, Sam Maltby, Matei Mandache, David Mestel, Jordan Millar, Kian Moshiri, Joseph Myers, Daniel Naylor, Jenny Owladi, Frankie Richards, Adrian Sanders, Amit Shah, Jack Shotton, Alan Slomson, Geoff Smith, Zhivko Stoyanov, Karthik Tadinada, David Vaccaro, Jenni Voon, Tommy Walker Mackay, Paul Walter, Zi Wang, Kasia Warburton, Dominic Yeo.

Mark distribution



The thresholds for qualification for BMO2 were as follows:

Year 13: 33 marks or more.

Year 12: 32 marks or more.

Year 11: 31 marks or more.

Year 10 or below: 29 marks or more.

The thresholds for medals, Distinction and Merit were as follows:

Medal and book prize: 33 marks or more.

Distinction: 21 marks or more.

Merit: 11 marks or more.

Question 1

Find three even numbers less than 400, each of which can be expressed as a sum of consecutive positive odd numbers in at least six different ways.

(Two expressions are considered to be different if they contain different numbers. The order of the numbers forming a sum is irrelevant.)

Solution

Consider runs of consecutive odd numbers with even sum. There must be an even number, $2k$, terms in such a run since the sum is even. The mean, $2u$, must be even since it must also be the median.

Therefore the sum is $4ku$. Notice that the largest odd number in the run is $2u + 2k - 1$ and the smallest is $2u - 2k + 1$. The smallest number must be positive so $k \leq u$. We therefore seek numbers K in the range 1 to 100 inclusive which have at least six factorizations $K = ku$ where $k \leq u$. Then the solutions are the numbers $N = 4K$.

To solve the problem we must provide three of the five possible values of K, which are $60 = 2^2 \times 3 \times 5, 72 = 2^3 \times 3^2, 84 = 2^2 \times 3 \times 7, 90 = 2 \times 3^2 \times 5$ and $96 = 2^5 \times 3$.

These give rise to five possible values of N, namely 240, 288, 336, 360, 384.

The quickest way to show that these values of K work is to recall that if $p_1, p_2, \ldots p_m$ are different prime numbers, then the different factors of

$$p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m} \quad \text{are} \quad p_1^{x_1} p_2^{x_2} \cdots p_m^{x_m}$$

where there are $a_i + 1$ options for each exponent x_i because $0 \leq x_i \leq a_i$, and so the number of positive integer divisors is

$$(a_1 + 1)(a_2 + 1) \cdots (a_m + 1).$$

Alternatively we could write out each factorisation N into six pairs of even factors or the decomposition of N into six sums of consecutive positive odd number explicitly.

The details are as follows:

$$\begin{aligned}
240 &= 2^4 \cdot 3 \cdot 5 \\
&= 119 + 121 = 2 \times 120 \\
&= 57 + \cdots + 63 = 4 \times 60 \\
&= 35 + \cdots + 45 = 6 \times 40 \\
&= 23 + \cdots + 37 = 8 \times 30 \\
&= 15 + \cdots + 33 = 10 \times 24 \\
&= 9 + \cdots + 31 = 12 \times 20
\end{aligned}$$

$$\begin{aligned}
288 &= 2^5 \cdot 3^2 \\
&= 143 + 145 = 2 \times 144 \\
&= 69 + \cdots + 75 = 4 \times 72 \\
&= 43 + \cdots + 53 = 6 \times 48 \\
&= 29 + \cdots + 43 = 8 \times 36 \\
&= 13 + \cdots + 35 = 12 \times 24 \\
&= 3 + \cdots + 33 = 16 \times 18
\end{aligned}$$

$$\begin{aligned}
336 &= 2^4 \cdot 3 \cdot 7 \\
&= 167 + 169 = 2 \times 168 \\
&= 81 + \cdots + 87 = 4 \times 84 \\
&= 51 + \cdots + 61 = 6 \times 56 \\
&= 35 + \cdots + 49 = 8 \times 42 \\
&= 17 + \cdots + 39 = 12 \times 28 \\
&= 11 + \cdots + 37 = 14 \times 24
\end{aligned}$$

$$\begin{aligned} 360 &= 2^3 \cdot 3^2 \cdot 5 \\ &= 179 + 181 = 2 \times 180 \\ &= 87 + ... + 93 = 4 \times 90 \\ &= 55 + ... + 65 = 6 \times 60 \\ &= 27 + \cdots + 45 = 10 \times 36 \\ &= 19 + \cdots + 41 = 12 \times 30 \\ &= 3 + \cdots + 37 = 18 \times 20 \end{aligned} \qquad \begin{aligned} 384 &= 2^7 \cdot 3 \\ &= 191 + 193 = 2 \times 192 \\ &= 93 + \cdots + 99 = 4 \times 96 \\ &= 59 + \cdots + 69 = 6 \times 64 \\ &= 41 + \cdots + 55 = 8 \times 48 \\ &= 21 + \cdots + 43 = 12 \times 32 \\ &= 9 + \cdots + 39 = 16 \times 24 \end{aligned}$$

Alternative

Rather than work with the average of the terms in each sum, we can simply recall that the sum of n terms of an arithmetic progression with first term a and common difference 2 is $\frac{1}{2}(2a + 2(n-1))n = n(a + n - 1)$. We note that a must be odd and n must be even, so we set $a = 2b + 1$ and $n = 2k$ with $b \geq 0$. Now we seek numbers $N \leq 400$ which can be written as $N = 4k(k + b)$ for at least six different pairs (k, b). We conclude as in the first solution.

Alternative

It is well-known that the sum of the first t positive odd integers is t^2. Thus a positive number is a sum of consecutive odd integers if, and only if, it is a positive difference of two squares. Therefore we are looking for even positive integers N of the form $N = u^2 - v^2 = (u - v)(u + v)$. We are given that N is even, which implies that u and v have the same parity, which in turn implies that both the factors $(u - v)$ and $(u + v)$ are even. This logic is reversible because if $0 < a < b$ are integers with $N = (2a)(2b)$, we can set $u = b + a$ and $v = b - a$.

Therefore we must find those even N which have at least 6 different factorizations $N = (2a)(2b)$ with $0 < a < b$, or after dividing by 4, we need to find positive integers less than 100 which have at least 6 factorizations. We conclude as in the first solution.

Markers' comments

There were many excellent solutions to this problem, with candidates providing detailed justifications for the three numbers they found. Yet more candidates found three correct numbers N, but did not fully justify that each satisfied the three conditions in the question namely that there are six ways to write N as a sum of an arithmetic progression with common difference 2, that all the terms in the sums are odd and that all the terms are positive. A common oversight was to include sums that do not satisfy one (or both) of the last two conditions.

For example, the number $192 = 16 \times 12$ can be written as a sum of 16 consecutive odd numbers with average value 12, but in this sum the first term is -3. On the other hand, the number $360 = 8 \times 45$ can be written as a sum of 8 positive numbers differing by 2 but they are even: $38 + 40 + \cdots + 52$. (Note that 360 actually works but many candidates failed to correctly explain why it does.) Another common incorrect number seen was 120; candidates giving this answer had often failed to deal correctly with either of the last two conditions.

How easy it is to avoid those traps depends on the exact approach taken to finding the numbers. Quite a large number of candidates ended up listing the possible sums: this was not essential, but is clearly a safe way of ensuring they all work. Starting from the fact that the first n odd numbers sum to n^2, as in the second alternative above, takes care of both the oddness and the positivity of the summands very easily.

Other approaches tended to observe that if we take $n = 2k$ numbers differing by 2 and call the first term a and/or the average term m, then the total, N, is given by

$$N = 2km = 2k(a + 2k - 1).$$

At this point, many candidates thought that just having six even *factors* was enough; however, since we need all terms to be positive, it is also required that $2k \leq m = a + 2k - 1$, so we actually need six distinct *factor pairs*, i.e. at least 11 factors.

The other common mistake, finding sums of even instead of odd numbers, arose when candidates included neither the fact that a had be odd, nor the fact that m had to be even, in their arguments.

Many candidates who found three correct numbers will be disappointed with the mark they received. The reason for this is that there were logical flaws in their explanations, mixing up necessary and sufficient conditions.

In effect, most candidates proved statements such as 'If a number can be written as six different sums of consecutive positive odds, then it can be divided by six different multiples of 4.' But this does not guarantee that every number which is divisible by six different multiples of 4 can be written in the required way. A complete solution needs to explain why the given argument is reversible, or describe how to get from a specific multiple of 4 to a sum of consecutive odd numbers. Hopefully this is a useful general lesson.

Question 2

One day Arun and Disha played several games of table tennis. At five points during the day, Arun calculated the percentage of the games played so far that he had won. The results of these calculations were exactly 30%, exactly 40%, exactly 50%, exactly 60% and exactly 70% in some order. What is the smallest possible number of games they played?

Solution

In order for Arun to be able to win 30% of N games, N must be a multiple of 10. The same holds for 70%, so the number of games must be at least 20.

But 20 is not possible: if it were, Arun would have to win either 30% of the first 10 games and then 70% of all 20, or 70% of the first 10 games and then 30% of all 20. In the first case the wins must go from $3/10$ to $14/20$ which requires 11 wins in 10 games; in the second case the wins go from $7/10$ to $6/20$ which requires -1 wins in 10 games.

So 30% and 70% alone cannot be achieved without reaching 30 games.

But 30 games is enough: for example, scores along the way could be 2/5, 3/10, 7/14, 12/20, 21/30 achieving 40%, 30%, 50%, 60%, 70% in that order.

Remark

There are many possible constructions that work with 30 games:

- 2/5, 3/10, 12/20, 21/30 with 50% won after 2, 4, 6, 14 or 16 games.
- 3/5, 3/10, 6/15, 21/30 with 50% won after 2, 4, 6 or 18 games.
- 2/5, 7/10, 9/15, 9/30 with 50% won after 2, 4, 6 or 18 games.
- 3/5, 7/10, 8/20, 9/30 with 50% won after 2, 4, 6, 14 or 16 games.

Alternative

Instead of doing the case of 20 games 'by hand', there is an algebraic alternative.

Suppose that Arun wins a of the first b games where $\frac{a}{b} = \frac{3}{10}$ and then wins m of the next n games so that $\frac{a+m}{b+n} = \frac{7}{10}$. (We can assume WLOG that 30% is achieved before 70% because otherwise we just count losses instead of wins). Since $n \geq m$, we have $\frac{a+n}{b+n} \geq \frac{7}{10}$. Multiplying up, we get $7(b+n) \leq 10(a+n) = 3b + 10n$, since $\frac{a}{b} = \frac{3}{10}$. This rearranges to $n \geq \frac{4}{3}b$. In particular, we need at least $\frac{7}{3}b$ games to achieve both 30% and a score above 70%. Since $b \geq 10$, we must have at least $70/3 > 20$ games in total. So to get exactly a score of 30% and a score of 70% we need to have a number of games that's a multiple of 10 and greater than 20, so at least 30.

Markers' comments

There were lots of good solutions to this question; there were also plenty of opportunities for partial marks. We were pleased to see that most candidates realized that you had to do two things to solve this problem: show both that you can do it in 30 games and that you cannot possibly do better.

The commonest way to go wrong was to think that the percentages had to appear in the order listed in the question. This gives a 40 game solution. If you fix an order for the percentages the problem gets much easier; you can solve it with a so-called 'greedy algorithm' where you always achieve the least percentage at the lowest possible score. In fact some candidates managed to get a solution by using this strategy over various different possible orderings. You can save time in such a solution by observations like "to get to any percentage, you must have at least 1 win and 1 loss, so we cannot be hurt by assuming 50% happens after 2 games", but you have to be careful to use provable facts and not heuristics like "It's obviously a good idea to use fractions with small denominators first". Plenty of candidates lost marks for vagueness like this.

Question 3

For each integer $0 \le n \le 11$, Eliza has exactly three identical pieces of gold that weigh 2^n grams. In how many different ways can she form a pile of gold weighing 2021 grams?

(*Two piles are different if they contain different numbers of gold pieces of some weight. The arrangement of the pieces in the piles is irrelevant.*)

Solution

Suppose there are $f(n)$ ways to choose n grams worth of gold. We begin by finding some small values of $f(n)$.

n	$f(n)$	collections of coins
1	1	{1}
2	2	{1,1}, {2}
3	2	{1,1,1}, {1,2}
4	3	{1,1,2}, {2,2}, {4}
5	3	{1,1,1,2}, {1,2,2}, {1,4}
6	4	{1,1,2,2}, {1,1,4}, {2,2,2},{2,4}

This suggests that $f(2k+1) = f(2k) = k+1$, which in turn suggests that $f(2021) = 1011$. It remains to prove that this pattern continues.

If $n = 2k$ there are either 0 or 2 pieces weighing one gram. In the first case we can halve each weight to give a way of choosing k grams, while in the second we can remove the two smallest coins and halve the remaining weights to give a way of choosing $k-1$ grams, thus $f(2k) = f(k) + f(k-1)$.

The fact that $f(2k+1) = f(k) + f(k-1)$ can be shown analogously, by removing the one or three coins weighing one gram and halving the remaining weights.

Alternatively we can observe that ways of choosing $2k+1$ grams correspond exactly to ways of choosing $2k$ grams: we simply add or remove a single one gram coin.

Having shown that $f(2k+1) = f(2k) = f(k)+f(k-1)$ we can prove that $f(2k+1) = f(2k) = k+1$ by induction using the following four steps.

$f(4k+3) = f(2k+1) + f(2k) = (k+1) + (k+1)$

$f(4k+2) = f(2k+1) + f(2k) = (k+1) + (k+1)$

$f(4k+1) = f(2k) + f(2k-1) = (k+1) + k$

$f(4k) = f(2k) + f(2k-1) = (k+1) + k.$

Thus 2021 grams of gold can be chosen in 1011 ways.

Alternative

Once we have established the recurrences $f(2k+1) = f(2k) = f(k) + f(k-1)$ we can avoid the induction and proceed 'by hand' as follows:

$$\begin{aligned}
f(2021) &= 1f(1010) + 1f(1009) \\
&= 1f(505) + 2f(504) + 1f(503) \\
&= 3f(252) + 4f(251) + 1f(250) \\
&= 3f(126) + 8f(125) + 5f(124) \\
&= 3f(63) + 16f(62) + 13f(61) \\
&= 19f(31) + 32f(30) + 13f(29) \\
&= 51f(15) + 64f(14) + 13f(13) \\
&= 115f(7) + 128f(6) + 13f(5) \\
&= 243f(3) + 256f(2) + 13f(1) \\
&= 499f(1) + 512f(0) \\
&= 1011
\end{aligned}$$

Remark

The way that the coefficients in each row add to make those in the row below, depends on the parity of the numbers in the first column. These, in turn, depend on the binary representation of 2021. This idea can be used to show that, in general, $f(n)$ can be found by taking the binary representation of n, removing the rightmost binary bit and adding 1.

Alternative

Let a_n be the number of ways of writing n as a sum of powers of 2 with each power appearing at most 3 times. We will work with the generating function of the sequence (taking $x \in (0, 1)$ so everything converges).

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n x^n &= \lim_{m\to\infty} \prod_{k=0}^{m} \left(1 + x^{2^k} + x^{2\cdot 2^k} + x^{3\cdot 2^k}\right) \\
&= \lim_{m\to\infty} \prod_{k=0}^{m} \frac{1 - x^{2^{k+2}}}{1 - x^{2^k}} \\
&= \lim_{m\to\infty} \frac{\left(1 - x^{2^{m+2}}\right)\left(1 - x^{2^{m+1}}\right)}{(1 - x)\left(1 - x^2\right)} \\
&= \frac{1}{(1 - x)\left(1 - x^2\right)} \underbrace{\lim_{m\to\infty} \left(1 - x^{2^{m+2}}\right)\left(1 - x^{2^{m+1}}\right)}_{=1\times 1} \\
&= \frac{1}{(1 - x)\left(1 - x^2\right)}
\end{aligned}$$

By differentiating $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ or using the binomial series, we get $\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n + 1)x^n$. Hence:

$$\begin{aligned}
\frac{1}{(1 - x)\left(1 - x^2\right)} = \frac{1 + x}{\left(1 - x^2\right)^2} &= (1 + x)\left(1 + 2x^2 + 3x^4 + \dots\right) \\
&= 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 \dots
\end{aligned}$$

This gives $a_{2n+1} = a_{2n} = n + 1$ as before and specifically, $a_{2021} = 1011$.

ALTERNATIVE

We can view Eliza's piles of gold as binary representations of 2021 where each digit is allowed to be as high as three. We can count the number of such representations by starting with the standard representation $2021 =_2 11111100101$ and working along it from left to right. (The first alternative solution essentially counts these expressions working from right to left). For each binary bit we may ask 'Can we increase this by reducing the value of the bit to the left?' Initially this feels like a 'Yes/No' question which would give a binary decision tree. However, the situation is more subtle: it is possible to increase a binary bit to be as high as 4 or even 5 and still obtain a valid sum, provided we reduce that bit appropriately in the next step. We can never have a bit of 6 or more since $6 \times 2^a > 3(2^a + 2^{a-1} + \cdots + 1)$.

Thus, as we work along 11111100101 we will have ten opportunities to increase the bit in question by 0, 2 or 4. We define $a_i(k)$ for $i = 0, 2, 4$ to be the number of decision sequences where the k^{th} bit is increased by i. These depend on the value of the $(k-1)^{\text{th}}$ or *prior* bit at that stage. If the prior bit is 5 (which happens in exactly $a_4(k-1)$ ways), then we must reduce it by 2, so the $a_4(k-1)$ contributes to $a_4(k)$ only. If the prior bit is 4 (also $a_4(k-1)$ ways), then we can reduce it by either one or two, so $a_4(k-1)$ contributes to both $a_4(k)$ and $a_2(k)$. Similarly, if the prior bit is 3 or 2 ($a_1(k-1)$ ways), then we can reduce it by zero, one or two, so $a_1(k-1)$ contributes to all three $a_i(k)$ values. If the prior bit is 1, we get a contribution (of $a_0(k-1)$) to $a_2(k)$ and $a_0(k)$, while if the prior bit is 0 we cannot reduce it, so we only get a contribution of $a_0(k-1)$ to $a_0(k)$. Putting all this together, we obtain the following table.

k	Bit in 2021	$a_0(k)$	$a_2(k)$	$a_4(k)$
1	1	1	0	0
2	1	1	1	0
3	1	2	2	1
4	1	4	4	3
5	1	8	8	7
6	1	16	16	15
7	0	32	32	31
8	0	64	63	63
9	1	127	126	126
10	0	253	253	252
11	1	506	505	505

We can discount the 505 sequences which require Eliza to use five 1g pieces, to obtain the final answer $506 + 505 = 1011$.

ALTERNATIVE

It is possible to construct a one-one correspondence between the numbers $0, 1, 2, \ldots, 1010$ and the legal decompositions of 2021 as follows.

Choose $0 \leq m \leq 1010$ and write it in binary.

Now write it in binary again.

Now write $2021 - 2m$ in binary.

Sum the three binary expressions without carries to obtain an expression for 2021 as a sum of powers of 2, each occurring at most three times.

This process is reversible, since we may start with a legal decomposition of 2021, remove a copy of each power of 2 occurring once or thrice, then take one copy of each power of two now occurring twice to recover the value of m.

Markers' comments

This question was found very difficult, and more marks were earned on both the subsequent questions. The main problem was that surprisingly few candidates thought to systematically investigate how many ways piles of 1 gram, 2 grams, 3 grams and so on could be formed. They therefore missed the clear pattern, and did not think to try an inductive approach.

Many candidates started by writing 2021 in binary and then considering which of the powers of two in this representation Eliza could split; unfortunately the majority failed to appreciate that the choices made at each stage are not independent (Eliza can split an 8 gram piece if and only if she first splits at least one 16 gram piece). This meant that multiplying numbers of options together was doomed to failure. Indeed, while this 'top down' approach can be made to work, it is certainly harder than what was intended.

Despite this, there were a number of excellent solutions, and it was pleasing to see all five methods outlined above being offered successfully.

The first alternative is the most accessible, but to score highly using a numerical approach like this requires an explanation as to *why* $f(2n) = f(n) + f(n-1)$ (as well as patience and accuracy).

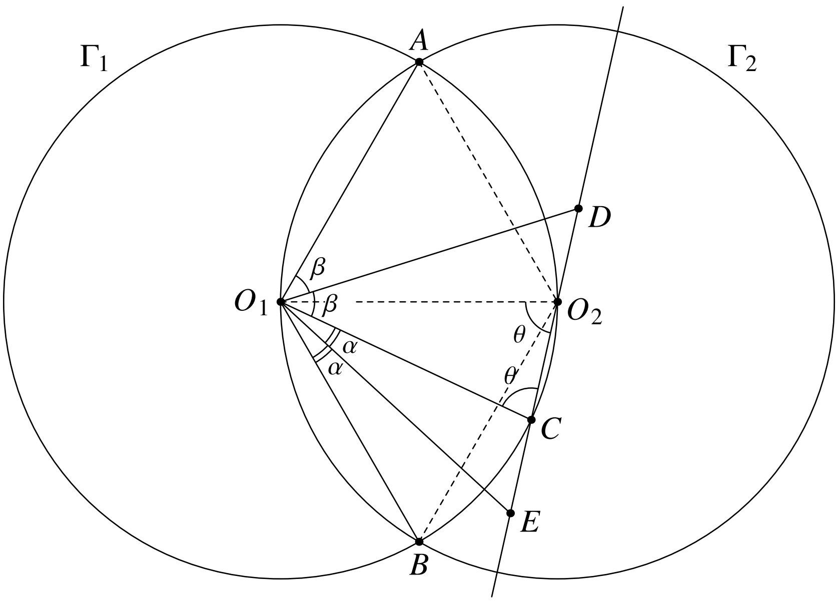
Question 4

Two circles Γ_1 and Γ_2 have centres O_1 and O_2 respectively. They pass through each other's centres and intersect at A and B. The point C lies on the minor arc BO_2 of Γ_1. The points D and E lie on the line O_2C such that $\angle AO_1D = \angle DO_1C$ and $\angle CO_1E = \angle EO_1B$. Prove that triangle DO_1E is equilateral.

(A minor arc of a circle is the shorter of the two arcs with given endpoints.)

Solution

It suffices to show that two of the angles in triangle DO_1E are equal to 60°.



The triangles AO_1O_2 and BO_1O_2 are both equilateral since their sides all equal the radius O_1O_2. They therefore have angles of 60°.

The question give us that $\angle BO_1E = \angle EO_1C$, which we will call α, and also that $\angle CO_1D = \angle DO_1A$, which we will call β. It is clear that $2\alpha + 2\beta = 120°$, so $EO_1D = \alpha + \beta = 60°$.

To complete the problem we must show that one of the other angles DO_1E is 60°.

There are a wide variety of possible approaches.

The triangle CO_1O_2 is isosceles since two of its sides are radii.

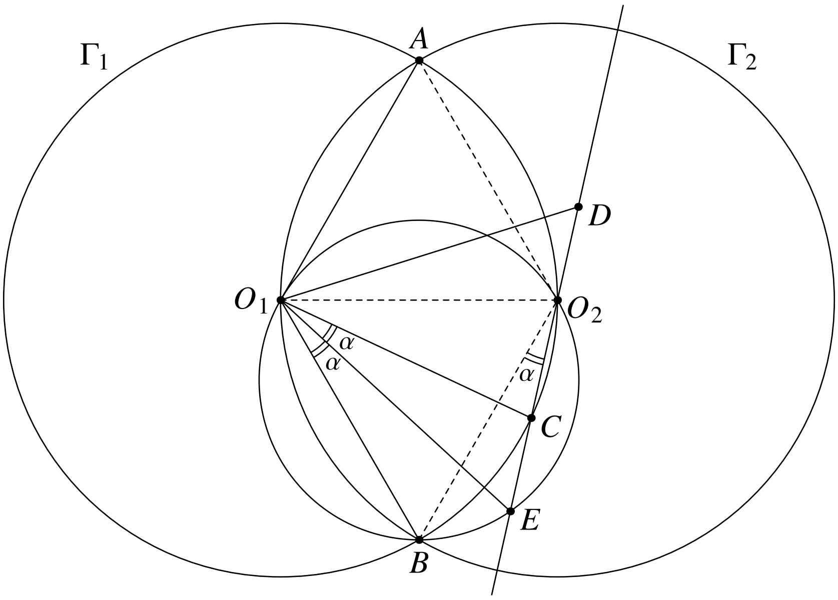
Therefore $\angle O_1O_2C = \angle O_2CO_1$ which we will call θ.

Now $2\beta = \angle O_2O_1A + \angle CO_1O_2 = 60° + (180° - 2\theta)$, so $\beta = 120 - \theta$.

Considering the sum of the angles in triangle DO_1C, we see that

$\angle O_1DC = 180° - \theta - (120° - \theta) = 60°$ as required.

Alternative

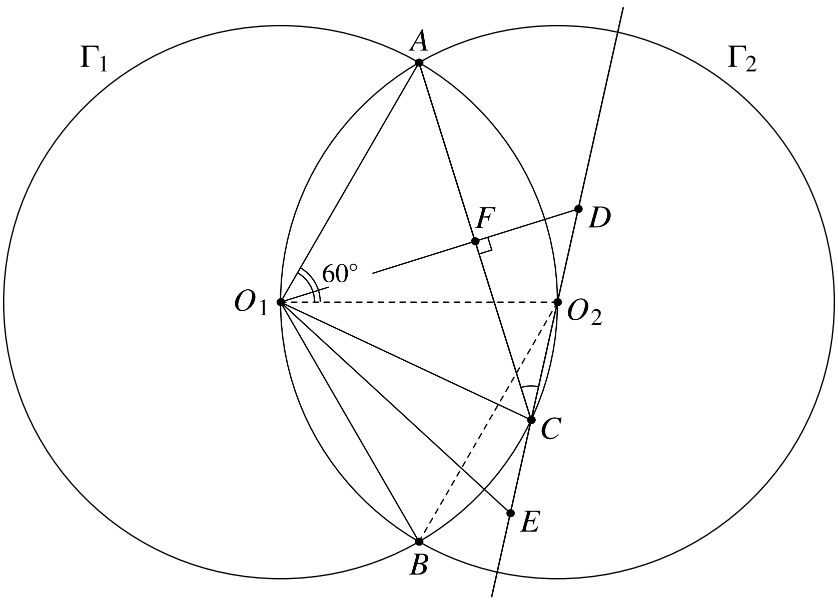


Using the fact that the angle at the centre is half the angle at the circumference, we see that $\angle BO_2C = \frac{1}{2}\angle BO_1C = \alpha$.

This shows that $\angle BO_1E = \angle BO_2E$ so BO_1O_2E is cyclic.

Now using angle in the same segment we see that $\angle O_2EO_1 = \angle O_2BO_1 = 60°$.

Alternative



Let O_1D meet AC at F. Since O_1F is the angle bisector of the isosceles triangle AO_1C, it is also the altitude, so $\angle O_1FA$ is a right angle.

Since the angle at the circumference is half the angle at the centre, we have that

$\angle O_2CA = \frac{1}{2}\angle O_1O_1A = 30°$.

Now considering the angles in triangle CDF gives

$\angle FDC = 180° - 30° - 90° = 60°$.

Remark

It is also possible to establish that DO_1E is equilateral by showing that $O_1D = O_1E$, this can be done by, for example, establishing that triangles O_1O_2D and O_1CE are congruent.

Remark

The condition that C lies on the minor arc O_2B is not essential to the problem. The result holds for nearly all C on the circle Γ_1. However, the wording of the problem avoids some small technical issues. In particular, if $C = O_2$ the result still holds provided we take the line O_2C to be the tangent to Γ_1 at O_2; if $C = B$ and $D = O_2$ the result only holds if we insist that the angles in the question are *directed* otherwise E can lie anywhere on O_2C and if C is diametrically opposite O_2 then the angles in the question are not defined.

Markers' comments

The first challenge on this problem was to draw a diagram. Many fell at this stage by assuming D and E must lie between or be coincident with O_2 and C, misinterpreting a line as a line segment.

Although most candidates then noticed that the two circles had the same radii, some missed the equilateral triangles and associated 60° angles, or failed to relate these to the angles in the problem. Candidates who did spot and use the equilateral triangles generally went on to prove that one angle in the target triangle was 60°.

There were multiple ways to proceed from here, and there were many successful solutions using congruence, cyclic quadrilaterals or simple angle chasing.

There were also a number of candidates who thought they had finished the problem, but were awarded low marks. Many of these used one or more of the following three points in their arguments: (i) the foot of the perpendicular from O_1 to DE, (ii) the point on both DE and the angle bisector of $\angle DO_1E$, (iii) the midpoint of DE. Since DO_1E turns out to be isosceles, these three points are, in fact, all the same. However, defining one of these points and assuming it has the properties of another in order to show that $O_1D = O_1E$ is a circular argument. Defining this point in terms of triangle O_2O_1C, which is clearly isosceles, led to correct solutions.

Question 5

An N-set is a set of different positive integers including a given positive integer N. Let $m(N)$ be the smallest possible mean of any N-set. For how many values of N less than 2021 is $m(N)$ an integer?

Solution

If $m(N) = m$ is an integer, then we may add m if it is missing, or remove m if it is present, without changing the mean. Moreover if m is minimal, then the N-set cannot contain an integer between m and N since removing it would reduce the mean. On the other hand, it must contain every integer below m since adding in any such integer would reduce the mean. It follows that

$$\frac{(1 + 2 + \ldots + m - 1 + N)}{m} = \frac{(1 + 2 + \ldots + m + N)}{(m + 1)} = m$$

This rearranges to $N = \frac{m(m+1)}{2}$, so $m(N)$ is an integer if and only if N is a triangle number.

Since $\frac{1}{2} \times 64 \times 63 = 2016$, there are 63 triangle numbers below 2021.

Alternative

Any N-set of size n with minimal mean must be $1, 2, \ldots, n - 1, N$, so consider the mean of such sets

$$f(n) = \frac{n(n - 1)/2 + N}{n} = \frac{n - 1}{2} + \frac{N}{n}.$$

So $f(n) - f(n - 1) = \frac{1}{2} - \frac{N}{n(n-1)}$ and $f(n + 1) - f(n) = \frac{1}{2} - \frac{N}{n(n+1)}$.

For minimal $f(n)$ we require $n(n - 1) \le 2N \le n(n + 1)$, giving $(n - 1)/2 \le N/n \le (n + 1)/2$. Hence $n - 1 \le f(n) \le n$. So $m(N)$ is an integer if and only if we have either equality, namely whenever $N = n(n \pm 1)/2$ is a triangle number. Now conclude as before.

Alternative

We prove that $m(T_k) = k$ for the triangle numbers as above and then observe that

$$m(N) = \frac{1}{k + 1}(1 + 2 + \ldots + k + N) > \frac{1}{k + 1}(1 + 2 + \ldots + k + (N - 1)) \ge m(N - 1)$$

so that $m(N)$ is a strictly increasing function. So $m(N)$ for non-triangle numbers must lie between integers.

Markers' comments

This question was attempted by many candidates, though more thought they had produced a full solution than actually had. On the basis of small examples, many candidates correctly conjectured that N has to be a triangle number. This was a good preliminary step, but not sufficient to gain any marks. To solve the problem we must show that if N is a triangle number, then $m(N)$ is an integer, and, conversely, that if $m(N)$ is an integer, then N is a triangle number. Many candidates only addressed one of these two assertions or were too vague in their reasoning.

Scripts that had clearly formed an equation describing exactly when $m(N)$ was an integer were marked generously, but candidates should remember that finding all the solutions to an equation always involves two logical steps, namely exhibiting the solutions and showing that there are no others. A variety of methods were used successfully, but most full solutions bounded $m(N)$ in terms of the second largest integer in an N-set of smallest mean.

A significant minority of candidates considered the function

$$f(n) = \frac{1 + 2 + \cdots + (n-1) + N}{n} = \frac{n-1}{2} + \frac{N}{n}$$

and then used calculus to find the minimum value of f. Unfortunately making this approach work is delicate. In particular, if $f(\alpha)$ is a minimum of f for some real α, and $f(n)$ is a minimal value of f when f's domain is restricted to the integers, then it is not necessarily the case that n and α are close together. This subtlety was missed by many candidates, and only a small proportion of those using calculus solved the problem successfully.

Question 6

Marvin has been tasked with writing down every list of integers with the following properties:

(i) The list contains 71 terms.

(ii) The first term is 1.

(iii) Every term after the first is equal to either the previous term, or the sum of all previous terms.

When Marvin is finished, how many of the lists will have a sum equal to 999,999?

SOLUTION

Each term after the first in one of Marvin's lists is equal either to the previous term, or to the sum of all previous terms. Let's label terms of the first type with a P for 'previous', and terms of the second type with an S for 'sum'. In this way from each of Marvin's lists of 71 numbers we generate a sequence of 70 labels, each a P or S. We may assume the sequence of labels begins with an S.

Now break the sequence of 70 labels into blocks, each of which consists of an S followed by some number of P's (possibly zero). Let $b_1, b_2, \ldots, b_n$ be the lengths of the blocks in order, so $b_1 + b_2 + \ldots + b_n = 70$. [As a miniature example, if the list was 1, 1, 2, 4, 4, 12; the corresponding sequence of labels would be S, S, S, P, S; and the lengths of the blocks would be $(b_1, b_2, b_3, b_4) = (1, 1, 2, 1)$.] Given the block lengths (b_i) with sum 70 we can recover the original list of integers: the block lengths allow us to write down the 70 S/P labels in order; then the original list of 71 is recovered uniquely by starting with a 1 and generating successive terms as the repetition of the previous term [P] or the sum of all previous terms [S].

Consider one of Marvin's lists $t_1, t_2, \ldots, t_{71}$. Suppose that for some k, the labels of the terms $t_k, \ldots, t_{k+b-1}$ form a block of length b. If the sum of the terms before t_k is T, then $t_k = t_{k+1} = \cdots = t_{k+b-1} = T$. So the sum of the terms of the list up to the $(k + b - 1)^{\text{th}}$ is $T + bT = (b + 1)T$. That is, the block of length b has the effect of multiplying the sum of the list by $b + 1$. Since the first term is always 1, the sum of all the terms in one of Marvin's lists is $(b_1 + 1)(b_2 + 1) \cdots (b_n + 1)$. We are interested in those lists for which $(b_1 + 1)(b_2 + 1) \cdots (b_n + 1) = 999,999 = 3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$ and $b_1 + b_2 + \cdots + b_n = 70$.

All the lists for which $n = 7$ and $b_1, \ldots, b_7$ is some permutation of $2, 2, 2, 6, 10, 12, 36$ satisfy this condition. There are $7!/3! = 840$ such lists. We claim there are no other possibilities. In particular, it suffices to show that for each i, the quantity $b_i + 1$ must be prime. Suppose not; then there is some sequence (b_i) of block lengths which satisfies both the multiplicative condition $\prod(b_i + 1) = 999,999$ and the additive condition $\sum b_i = 70$, and which contains a block of length b_j satisfying $b_j + 1 = ab$ for $a, b > 1$. Then we can replace b_j by $a - 1, b - 1 > 0$ to obtain a new sequence (b'_i) of block lengths satisfying the required product condition and with smaller sum (since $(ab - 1) - (a - 1) - (b - 1) = (a - 1)(b - 1) > 0$). Repeated decomposition of composite terms results in the sequence $2, 2, 2, 6, 10, 12, 36$ by the uniqueness of prime factorization of $999,999$. But this means that the sum of terms in the sequence (b_i) was greater than 70, a contradiction.

Alternative

As before, the block lengths must satisfy the equations $(b_1 + 1)(b_2 + 1) \cdots (b_n + 1) = 999,999 = 3 \cdot 3 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$, and $b_1 + b_2 + \ldots + b_n = 70$. Consider the prime factors 7, 11, 13 and 37. No bracketed term $(b_i + 1)$ in the product can be divisible by two of those primes, for in that case we would have $b_i \geq 76 > 70$. And in fact four of the bracketed terms must be exactly 7, 11, 13 and 37, else the sum of the b_i would be at least $13 + 10 + 12 + 36 > 70$. So some four blocks have lengths 6, 10, 12 and 36 respectively. The remaining blocks have total length $70 - 36 - 12 - 10 - 6 = 6$. And each of these remaining blocks must have length one less than a power of 3 because the only prime factors of 999,999 still to be accounted for are $3 \cdot 3 \cdot 3$. So they are three blocks of length 2. Therefore, the block lengths are 36, 12, 10, 6, 2, 2, 2 in some order. The solution is completed as before.

Markers' comments

For its position on the paper, this question was found approachable. Many candidates had the excellent intuition to look at blocks of repeated terms in the sequence, and a significant proportion of those were able to derive the formula for the sum of one of Marvin's lists in terms of the block lengths.

Candidates found it much harder to write down the next stage of the proof: showing that all the block lengths must be of length 'prime−1'. Quite a few scripts gave the 'correct answer' of 840, but it was vital to show that all other possibilities fail, so just getting 840 did not mean the problem was essentially solved. Only a handful of solutions addressed this issue well enough to score close to full marks.

Another common problem in otherwise good scripts was that, having turned the problem into algebra and counted the solutions to a relevant pair of equations, candidates often neglected to check that each solution to the equations corresponded to precisely one of Marvin's lists. Failing to engage with this subtlety attracted a 1 mark penalty.

Many candidates tried to make a 'binary' tree diagram of choices; this turned out not to be that helpful (most of the choices lead to sums that do not give 999,999). Others tried to write down all the possible values of early numbers in the sequence, but many did not think carefully about the rules, which for example showed that the fifth term could not equal either 5 or 7.