May 2, 2019
Language: English

Problem 1.
Let $\mathbb{P}$ be the set of all prime numbers. Find all functions $f : \mathbb{P} \to \mathbb{P}$ such that

$$f(p)^{f(q)} + q^p = f(q)^{f(p)} + p^q$$

holds for all $p, q \in \mathbb{P}$.

Problem 2.
Let a, b, c be real numbers, such that $0 \leq a \leq b \leq c$ and $a + b + c = ab + bc + ca > 0$. Prove that $\sqrt{bc}(a + 1) \geq 2$. Find all triples (a, b, c) for which equality holds.

Problem 3.
Let ABC be an acute triangle. Let X and Y be two distinct interior points of the segment BC such that $\angle CAX = \angle YAB$. Suppose that:

1) K and S are the feet of perpendiculars from B to the lines AX and AY respectively;
2) T and L are the feet of perpendiculars from C to the lines AX and AY respectively.

Prove that KL and ST intersect on the line BC.

Problem 4.
A grid consists of all points of the form (m, n) where m and n are integers with $|m| \leqslant 2019$, $|n| \leqslant 2019$ and $|m| + |n| < 4038$. We call the points (m, n) of the grid with either $|m| = 2019$ or $|n| = 2019$ the *boundary points*. The four lines $x = \pm 2019$ and $y = \pm 2019$ are called *boundary lines*. Two points in the grid are called *neighbours* if the distance between them is equal to 1.

Anna and Bob play a game on this grid.

Anna starts with a token at the point $(0, 0)$. They take turns, with Bob playing first.

1) On each of his turns, Bob *deletes* at most two boundary points on each boundary line.

2) On each of her turns, Anna makes exactly three *steps*, where a *step* consists of moving her token from its current point to any neighbouring point, which has not been deleted.

As soon as Anna places her token on some boundary point which has not been deleted, the game is over and Anna wins.

Does Anna have a winning strategy?

Time allowed: 4 hours and 30 minutes.
Each problem is worth 10 points.