Saturday, July 9, 2022

Problem 1. For every integer $n \geq 1$ consider the $n \times n$ table with entry $\left\lfloor \frac{ij}{n+1} \right\rfloor$ at the intersection of row i and column j, for every $i = 1, \ldots, n$ and $j = 1, \ldots, n$. Determine all integers $n \geq 1$ for which the sum of the n^2 entries in the table is equal to $\frac{1}{4}n^2(n-1)$.

Problem 2. Let $ABCD$ be a quadrilateral inscribed in a circle Ω. Let the tangent to Ω at D intersect the rays BA and BC at points E and F, respectively. A point T is chosen inside the triangle ABC so that $TE \| CD$ and $TF \| AD$. Let $K \neq D$ be a point on the segment DF such that $TD = TK$. Prove that the lines AC, DT and BK intersect at one point.

Problem 3. Consider a $3m \times 3m$ square grid, where m is an integer greater than 1. A frog sits on the lower left corner cell S and wants to get to the upper right corner cell F. The frog can hop from any cell to either the next cell to the right or the next cell upwards.

Some cells can be sticky, and the frog gets trapped once it hops on such a cell. A set X of cells is called *blocking* if the frog cannot reach F from S when all the cells of X are sticky. A blocking set is *minimal* if it does not contain a smaller blocking set.

(a) Prove that there exists a minimal blocking set containing at least $3m^2 - 3m$ cells.

(b) Prove that every minimal blocking set contains at most $3m^2$ cells.

Note. An example of a minimal blocking set for $m = 2$ is shown below. Cells of the set X are marked by letters x.

					F
x	x				
		x			
			x		
				x	
S		x			