Saturday, July 13, 2019

Problem 1. Determine all pairs (n, k) of distinct positive integers such that there exists a positive integer s for which the numbers of divisors of sn and sk are equal.

Problem 2. Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. On each turn, Sisyphus chooses any nonempty square, say with k stones, takes one of those stones and moves it to the right by at most k squares (the stone must stay within the board). Sisyphus' aim is to move all n stones to square n.

Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns.

(As usual, $\lceil x \rceil$ stand for the least integer not smaller than x.)

Problem 3. Triangle ABC has circumcircle ω and incentre I. A line ℓ intersects lines AI, BI, and CI at points D, E, and F, respectively, distinct from points A, B, C, and I. The perpendicular bisectors of segments AD, BE, and CF determine a triangle Θ.

Show that the circumcircle of triangle Θ is tangent to ω.

Language: English *Time: 4 hours and 30 minutes*
Each problem is worth 7 points